

# Dynamic Valuation of Delinquent Credit-Card Accounts\*

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## Abstract

This paper introduces a dynamic model of the stochastic repayment behavior exhibited by delinquent credit-card accounts. Based on this model, we construct a dynamic collectability score (DCS) which estimates the account-specific probability of collecting a given portion of the outstanding debt over any given time horizon. The model integrates a variety of information sources, including historical repayment data, account-specific, and time-varying macroeconomic covariates, as well as scheduled account-treatment actions. Two model-identification methods are examined, based on maximum-likelihood estimation and the generalized method of moments. The latter allows for an operational-statistics approach, combining model estimation and performance optimization by tailoring the estimation error to business-relevant loss functions. The DCS framework is applied to a large set of account-level repayment data. The improvements in classification and prediction performance compared to standard bank-internal scoring methods are found to be significant.

*Keywords:* Account valuation; consumer credit; collectability scoring; credit collections; GMM estimation; maximum-likelihood estimation; operational statistics; self-exciting point process.

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# 1 Introduction

The efficient collection of outstanding debt from defaulted credit-card accounts is mission-critical for many financial institutions. By the end of 2012, the total revolving consumer credit in the United States amounted to approximately \$603 billion (Federal Reserve Bank H.8, January 2013). During that year about \$17 billion in credit-card debt was placed in collections, driven by an average delinquency rate of 2.88%.<sup>1</sup> With such amounts at stake, even a small improvement in the effectiveness of debt collections would imply a significant bottom-line impact. For example, an industry-wide improvement of 1% in overall collection return could produce a gain in excess of \$170 million, thus decreasing banks’ cost of capital, which in turn would lead to more lending and easier access to credit for consumers.<sup>2</sup>

This paper introduces a dynamic model of consumer repayment behavior on overdue, so-called “delinquent,”<sup>3</sup> credit-card accounts placed in collections. Both the timing and the amount of repayments are assumed to be random. The repayment amounts are correlated and their distributions are obtained empirically. Each repayment influences the timing distribution of the subsequent repayments. This stochastic feedback element captures both the account holder’s willingness-to-repay and ability-to-repay the outstanding balance, which in turn provides valuable insight about the actions banks should take to maximize the expected collections revenue. In addition, the timing distribution of the repayments also depends on the account holder’s demographic characteristics, the account attributes, the current economic outlook, as well as planned account-treatment actions. The model is used to construct a probability distribution for the collectability (i.e., repayment likelihood) of a delinquent account. The resulting account-specific *dynamic collectability score* (DCS) estimates, at any time in the collections process, the probability of collecting a given percentage of the outstanding balance over a desired time horizon. The model and scoring technique are tested using a large set of account-level repayment data and two different model-estimation methods—maximum likelihood and prediction-error minimization. The improvements in classification and prediction performance compared to standard bank-internal scoring methods are significant. In addition, important insights are obtained for the pricing of delinquent debt in terms of commission rates offered to collection agencies.

There are four main methodological contributions. First, the model constructed in this paper represents, to the best of our knowledge, the only continuous-time stochastic approach to the prediction of repayment processes, leading to account-specific dynamic collection forecasts. The forecasts are conditional on the repayment history, observed covariates such as FICO score of the account holder, relevant macroeconomic data (e.g., prime rate), and a given

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<sup>1</sup>The estimate is based on the charge-off and delinquency rates on loans and leases at commercial banks, reported regularly by the Federal Reserve Bank.

<sup>2</sup>According to Sufi (2009), over 80% of bank financing extended to public firms is in the form of revolving lines of credit. Moreover, as detailed by Blanchflower and Evans (2004), credit cards play a major role in financing small businesses.

<sup>3</sup>Accounts with outstanding balances that are more than 30 days past due are considered *delinquent* and entered in the collections process (this definition applies to our dataset in Sec. 5; the details are institution-specific). The *collections process* consists of several collection phases often conducted by different outside collection agencies at commission rates that increase after each phase transition of the account to a different agency. The management of the overall collections process and the ownership of the delinquent account generally remain with the originating bank until portfolios of ‘uncollectable’ dead-reserve accounts are eventually sold off (e.g., via auctions).

sequence of account-treatment actions. Second, the DCS-based predictions about account holder’s repayment behavior have—in contrast to many bank-internal scoring systems— intrinsic meaning in terms of concrete repayment probabilities. The flexibility of the DCS, with respect to prediction horizon, repayment threshold, and timing, presents a significant improvement over extant collectability scores, which tend to be static and focused on very stylized predictions such as distinguishing “payers” from “non-payers”. Third, the model lays the foundation for account-treatment optimization, as it can be used to forward-simulate repayment-path realizations conditional on treatment actions. Fourth, the DCS enables the collections manager to obtain estimates for the expected net value of a delinquent credit-card account and an estimate of account-specific repayment activity in terms of number of repayments over a given prediction horizon.

From a practical viewpoint, the results developed here allow for a more precise prediction of repayment behavior. With an improved understanding of the sensitivity of the repayment process to covariates and treatment actions (in terms of the identified model parameters), it is possible to (at least locally) maximize the collection yield by optimizing over the available actions. In addition to producing a noticeable increase in collection yield, a rigorous model can help develop a realistic assessment of the risk caused by account delinquency and improve underwriting through a better choice of account parameters. Moreover, the valuation method implied by the model allows for an effective computation of “loss given default” (LGD),<sup>4</sup> as defined by the Basel II accords (Basel Committee on Banking Supervision 2004), which feeds into the determination of banks’ capital-reserve requirements. The valuation method can also be used for pricing portfolios of dead-reserve accounts for debt-leasing purposes. Lastly, our model enables the bank to establish effective performance-based collection guidelines.

## 1.1 Literature Review

*Optimization of Credit Collections.* Mitchner and Peterson (1957) were among the very first to study the credit-collections problem from an operations perspective. They noted that the empirical loss distribution is usually bimodal,<sup>5</sup> corresponding to the separate repayment behavior of “skips” (contact not available) versus “nonskips” (contact available), and focused on the optimal length of account treatment (which they termed “pursuit duration”) as a stopping problem. For this, Mitchner and Peterson were primarily concerned with distinguishing payers from non-payers, assuming full “conversion” in the case of payment, emphasizing that “[i]n a fully rigorous treatment, the entire sequence of intervals between payments, as well as the payment amounts, should be taken into account as part of the overall stochastic process” (p. 537).

Subsequent contributions to the credit-collections problem, as laid out by Mitchner and Peterson, were sparse. Rather than focusing on collections, credit-scoring models were developed for underwriting new accounts (Bierman and Hausman 1970; for a survey see Rosenberg and Gleit 1994) or for estimating the time-to-default (Bellotti and Crook 2009). The collections problem did not appear on the academic ‘radar screen’ until, in 2004, the Basel II accords put a spotlight on the capital-reserve requirements induced by the in-aggregate

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<sup>4</sup>The LGD is the percentage of the value of an asset that is lost in the event of a default.

<sup>5</sup>More recently, Schuermann (2005) also emphasizes bimodality as a critical feature of recovery distributions for defaulted consumer loans.

quite substantial loss given default (LGD) in consumer credit (Basel Committee on Banking Supervision 2004).<sup>6</sup> Thomas et al. (2005) provide a survey of the few notable consumer-credit models up to that point. Our approach to the credit-collections problem implements the Mitchner-Peterson program by viewing account-repayment paths as realizations of self-exciting point processes, conditional on observations of repayment history, account-specific and macroeconomic covariates, as well as past and scheduled future account-treatment actions. In a discrete-time framework, Almeida Filho et al. (2010) formulate a Markov decision problem for the optimal timing of a given sequence of account-treatment actions, which, however, is not conditioned on detailed account-specific repayment behavior. While the model in our paper supports a similar goal (namely, the optimization of account-treatment actions), it does strive for higher resolution through the estimation of a nonlinear dynamic model in continuous time and the explicit use of account-specific features to optimize treatment based on action-contingent forecasts of repayment behavior.<sup>7</sup>

*Corporate vs. Consumer Debt.* Consumer debt-repayment behavior differs from the corresponding corporate behavior in the determinants for delinquency as well as in the relative absence of market-based indicators such as bond prices (Gross and Souleles 2002; Duygan-Bump and Grant 2009). Nevertheless, the repayment process of overdue consumer debt bears similarities to the cash-flow stream of an index corporate default swap (CDS),<sup>8</sup> since both consist of random payments at random times. Unlike an index CDS, for which the loss distribution of the constituent loans are often assumed to be independent and identically distributed (i.i.d.) (see, e.g., Azizpour et al. 2011), in the case of delinquent consumer loans, repayment amounts are correlated. Moreover, the valuation of delinquent consumer loans is primarily useful if it is made at the account level, since return cash flows are subject to the issuers' collections practices.

This paper builds on the literature about corporate default risk. It extends the transform analysis developed in Duffie et al. (2000) to affine jump-diffusion processes that also include deterministic jump terms. Errais et al. (2010) apply this type of analysis to a family of self- and cross-excited point processes to capture the clustering behavior of corporate defaults. The idea of cross-excitation was introduced in finance by Jarrow and Yu (2001) through the concept of counterparty risk, where the default of one firm can affect the default of other firms. The results were extended by Collin-Dufresne et al. (2004) for the pricing of credit derivatives using an affine jump-diffusion model. The model presented here is related to Duffie et al. (2007) who use time-series dynamics of firm-specific and macroeconomic covariates to predict corporate defaults. It also extends Errais et al. (2010) by including an additional term that acts as the control variable modeling banks' collection strategy, and thus allows for account-treatment optimization.

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<sup>6</sup>Altman et al. (2005) provide a collection of papers on the determination of LGD. In the context of credit-card debt collections these discussions are not fully relevant, for the collections problem exists only because default has already occurred in the past. Given the bank's exposure at default (EAD), it is the focus of our paper to dynamically estimate the LGD conditional on a repayment history and other covariates, for a given time horizon and repayment threshold.

<sup>7</sup>The optimization of the credit-collections process based on our stochastic model of the repayment process is treated in a companion paper (Chehrazi et al. 2014).

<sup>8</sup>An index CDS can be viewed as a loss-insurance contract on a portfolio of corporate loans.

*Equilibrium Models of Consumer Credit with Default.* The problems of credit scoring and individual consumers’ default decisions (see, e.g., Efrat 2002) have been studied, to some extent, in the economics literature. This work focuses on a description of the lender-borrower interaction in the unsecured consumer-credit market to study the welfare implications of various regulatory policies. Chatterjee et al. (2008) analyze an equilibrium model in which credit scoring is used as an index encapsulating the borrower’s reputation and creditworthiness. Ausubel (1999), Chatterjee et al. (2007), and Cole et al. (1995), among others, use game-theoretic models with asymmetric information to describe the notion of credit scoring as a screening/signaling problem, in an attempt to explain the dynamics of the market for unsecured debt. Credit scoring is also studied in the statistics literature, where different classification models have been used to rank consumers based on their creditworthiness and probability of default. For example, Bierman and Hausman (1970), Boyes et al. (1989), Hand and Henley (1996), and West (2000) use different statistical methods, including neural networks, discriminant analysis, and decision trees to distinguish between “bad” and “good” borrowers. Bellotti and Crook (2009) include macroeconomic covariates (such as interest rate and the FTSE All-Share Index) to build a pre-default score based on survival analysis. Despite several studies focusing on estimating pre-default repayment probabilities (Banasik et al. 1999; Lopes 2008), the extant literature does not consider the estimation of post-default repayment probabilities that is the focus of this paper.

*Application and Estimation of Point Processes.* Bartlett (1963) originated the systematic study of departures of (stationary) point processes from the standard Poisson processes (Cox and Isham 1980) using autocorrelation functions and spectral analysis. In many practical applications, such as earthquake prediction (Ogata 1998), an important phenomenon is data clustering which naturally lends itself to modelling via self-exciting point processes (Hawkes 1971), incorporating a stochastic feedback element in the evolution of the intensity process. Relevant for our model of consumer repayment behavior is the class of affine point processes (Errais et al. 2010) in which the stochastic differential equation for the evolution of the intensity is linear and can accommodate jump-diffusion processes that we use to incorporate the information derived from covariates and account-treatment actions. Applications of self-exciting point processes in the literature include clustering of aircraft hijacking events (Holden 1986), criminal behavior (Mohler et al. 2011), contagion in fixed-income financial markets (Giesecke and Kim 2011), and conversion behavior in response to online advertising (Xu et al. 2014). The estimation of point processes using maximum-likelihood methods was pioneered by Rubin (1972) and Ogata (1978). Giesecke and Schwenkler (2012) examine likelihood estimators for affine point processes when the explanatory factors are not completely observed. We apply likelihood estimation in conjunction with a Cramér-von Mises error criterion (Lehmann and Romano 2005). For the credit-collections problem, we show in Secs. 4 and 5 that identifying the model parameters based on the generalized method of moments (GMM) can provide a better alternative to maximum-likelihood estimation (MLE), much in the spirit of the recent advances in operational statistics (Besbes et al. 2010).

## 1.2 Outline

Sec. 2 introduces the probabilistic structure of the repayment process, together with the main technical assumptions. In Sec. 3, we determine the probability of repayment and derive the DCS credit-scoring system as well as a formula for the expected present value of a delinquent credit-card account. Sec. 4 is concerned with the estimation of the parameters for the repayment and covariate processes. In Sec. 5, we apply the model and the results developed earlier and compare the predictive performance to that of standard bank-internal methods. In addition, we illustrate the optimization of treatment actions. Sec. 6 concludes.

## 2 Model

Consider a delinquent credit-card account placed in collections. The information that may be used to predict future account repayment activity stems from a variety of sources. First, the account is characterized by a set of attributes, including the credit limit, current outstanding balance, and annual percentage rate (APR). Second, the bank has account-holder-specific data such as FICO score, annual household income, mortgages, personal bankruptcy events, as well as credit limits and utilization rates for all issued credit cards (including those from other institutions). Third, the bank observes covariates which may be relevant for an entire group of accounts such as market and price indices, prime rate, and unemployment data. Last, the bank has access to the repayment history leading up to the current outstanding balance; the available information also includes past and scheduled future account-treatment actions such as letters, calls, settlement offers, repayment schedules, and legal actions.

A functional model of the repayment process should incorporate these heterogeneous sources of information. For this, the stochastic sequence of repayments is described by a point process, whose conditional arrival rate (intensity) is driven by past repayment events (a jump process), a vector of time-varying covariates (a Markov process), and a sequence of account-treatment actions (a deterministic jump process). The time-invariant covariates, which often include account-specific data, are used to determine the parameters for the evolution of the intensity, except for its long-run steady-state, which is obtained from the process of time-varying covariates. The latter often includes information which pertains to an entire group of accounts (e.g., in the form of macroeconomic indicators) and is therefore referred to as a group-covariate process. In what follows, we introduce the elements of the repayment process, before detailing the structure of the associated intensity and covariate processes.<sup>9</sup>

### 2.1 Repayment Process

A delinquent account with outstanding balance  $B_0 > 0$ , placed in the collections process at time  $t = 0$ , is credited with the repayment amounts  $Z_i$  at times  $T_i > 0$ , for  $i \in \{1, 2, \dots\}$ , until the balance is paid in full. We assume that the  $T_i$  are stopping times with  $T_i < T_{i+1}$ ,

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<sup>9</sup>A summary of the notation is provided in Appendix B.

and that the  $Z_i$  are nonnegative random variables.<sup>10</sup> For an outstanding balance  $B_i$  after the  $i$ -th repayment  $Z_i$ , denote by  $R_i = Z_i/B_{i-1}$  the corresponding relative repayment, so

$$Z_i = B_0 \prod_{k=1}^{i-1} (1 - R_k) R_i, \quad i \geq 1. \quad (1)$$

The random sequence  $N = (T_i, R_i, i \geq 1)$ , referred to as “repayment process,” characterizes the account’s repayment behavior. We assume that the  $R_i, i \geq 1$ , are i.i.d. Note that notwithstanding this assumption, the absolute repayments ( $Z_i$ ) are correlated. For example, as soon as an outstanding balance is paid in full ( $R_{i'} = 1$ , or equivalently  $Z_{i'} = B_{i'-1}$ , for some  $T_{i'}$ ), subsequent repayments generated by the model will be zero while relative repayments remain i.i.d. ( $Z_i = B_i R_i = 0 \cdot R_i$ , for  $i > i'$ ). Intuitively, the i.i.d. assumption is consistent with the “regenerative” structure of the repayment process, in the sense that taking into account the effects of a repayment and normalizing the remaining outstanding balance to one produces a structurally identical process. The distribution of the  $R_i$  is taken as exogenous and is estimated empirically. This distribution, together with Eq. (1), specifies a probability distribution for the repayment amounts ( $Z_i, i \geq 1$ ), whose fit is evaluated using the Empirical Generalized Runs (EGR) test in Sec. 5.

## 2.2 Intensity Process

We denote the counting process associated with  $(T_i, i \geq 1)$  by  $\mathcal{N}_t$ . This process is uniquely identified by an *intensity process*  $\lambda_t$ , which quantifies the conditional arrival rate of repayment events. The dynamics of the intensity process are described by a stochastic differential equation (SDE). Specifically, we assume that  $\lambda_t$  solves

$$d\lambda_t = \kappa(y)(\lambda_\infty(X_t) - \lambda_t) dt + \delta_1^\top(y) dJ_t + \delta_2^\top(y) da_t, \quad t \in \mathbb{R}_+, \quad (2)$$

where  $X_t$  is a Markov process with values in  $\mathcal{D} \subset \mathbb{R}^{n_x}$  (containing dynamic macroeconomic indicators),  $y \in \mathbb{R}_+^{n_y}$  is a constant vector (containing time-invariant account attributes),  $J_t$  is a two-dimensional jump process (describing the repayments), and  $a_t$  is a deterministic, nondecreasing, right-continuous, piecewise constant function, with values in  $\mathbb{R}^{n_a}$  (containing scheduled account-treatment actions). The functions  $\lambda_\infty(\cdot)$  and  $(\kappa, \delta_1, \delta_2)(\cdot)$  are assumed to be nonnegative-valued and affine.

To better understand the evolution of the intensity process  $\lambda_t$ , consider first the special case where  $(\delta_1, \delta_2) = 0$  and  $\lambda_\infty$  is constant. Then the jump terms in Eq. (2) vanish, and the intensity process is described by a deterministic law of motion,

$$\lambda_t = \lambda_\infty + (\lambda_0 - \lambda_\infty)e^{-\kappa t}, \quad t \in \mathbb{R}_+,$$

where  $\lambda_0 \geq 0$  is a given initial value. In this setting,  $\lambda_t$  reverts to its long-run steady-state  $\lambda_\infty$  at the rate  $\kappa$ . A similar intuition applies when jump terms are present and  $\lambda_\infty$  is not constant. The change in the intensity caused by the jump process  $J_t$  or account treatment

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<sup>10</sup>All random variables are defined with respect to a complete probability space  $(\Omega, \mathcal{F}, P)$  together with an information filtration  $\mathbb{F} = \{\mathcal{F}_t : t \in \mathbb{R}_+\}$ , satisfying the usual hypotheses (Protter 2005). Random variables (except for Greek letters) are generally capitalized with realizations appearing in lower case.

actions  $a_t$  dissipates at the instantaneous rate  $\kappa$  as  $\lambda_t$  reverts to its long-run steady-state  $\lambda_\infty$ . The long-run steady-state depends on the dynamic group-covariate process  $X_t$  which typically comprises macroeconomic observables (e.g., inflation rate, unemployment rate, CPI, and GDP). Hence, the better the economic outlook in terms of group covariates, the greater is the chance of recovering the outstanding debt in the long-run. In our formulation, the tendency to observe larger repayments in a booming economy is mapped by the model to a probabilistic increase in the number of repayments. Conversely, a tendency to observe smaller repayment amounts in an economic downturn translates into a decreased repayment likelihood. Therefore, without loss of generality, the repayment amounts  $(Z_i, i \geq 1)$  are assumed to be independent of  $X_t$ .

The jump process  $J_t = [\mathcal{N}_t, \mathcal{R}_t]^\top$  in the SDE (2) is an equivalent representation of the repayment process  $N$ , where  $\mathcal{N}_t = \sum_i \mathbf{1}_{\{T_i \leq t\}}$  and  $\mathcal{R}_t = \sum_i R_i \mathbf{1}_{\{T_i \leq t\}}$ . The counting process  $\mathcal{N}_t$  is independent of the repayment amount  $Z_i$ , reflecting the account holder’s *willingness-to-repay*. The mark process  $\mathcal{R}_t$  depends on the repayment amounts  $Z_i$ , indicating the account holder’s *ability-to-repay*. The difference in the two dimensions of the repayment process ( $\mathcal{N}_t$  vs.  $\mathcal{R}_t$ ) can be appreciated by considering the somewhat extreme (yet not too unlikely) event when a relative repayment vanishes ( $R_i = 0$ ), corresponding to a bounced check. Such an event signals the account holder’s high willingness to repay her debt, but at the same time, it also reveals her low ability to do so.

REMARK 1. The arrival rate  $\lambda_t$  of the counting process  $\mathcal{N}_t$  depends on the history of  $\mathcal{N}_t$  itself through *both* components of  $J$ . This *self-excitation* feature is useful for modelling stochastic phenomena with clustering effects, where the past arrival of events affects the arrival of future events. In the context of debt collection, the frequency and amount of past repayments convey information about the account holder’s willingness-to-repay ( $\mathcal{N}_t$ ) and ability-to-repay ( $\mathcal{R}_t$ ). Incorporating this information into the dynamics of the intensity process naturally leads to a self-exciting point process. Viewed from this perspective, our dynamic repayment model follows the Errais et al. (2010) form. However, a key difference between their model and ours is the presence of a deterministic jump process describing scheduled account-treatment actions. This enables us to probabilistically attribute the collections outcome over a given horizon to different treatment actions taken in the past, establishing a framework for account-treatment optimization (Chehrazi et al. 2014).

## 2.3 Covariate Process

We distinguish account-specific covariates that drive the fixed effects for any given account, and group covariates that determine random effects for a set (or “group”) of similar accounts.

### 2.3.1 Account-Specific Covariates

The account-specific covariates are summarized by the nonnegative, time-invariant vector  $y$  of dimension  $n_y$ . They include attributes such as credit limit, outstanding balance, interest rate, as well as information on the account holder, such as FICO score, mortgage payments, and status of other credit cards (current/past due/written off). This data is either held by the bank or made available by various credit-rating agencies, usually on a monthly basis.



To separate the estimation of the dynamics of these covariates from the estimation of the account’s repayment behavior, we assume them to be fixed at their current values. Modelling the covariates as a time-varying process would increase the parameter space and likely deteriorate the out-of-sample performance due to overfitting (Hastie et al. 2009). This is also consistent with much of the credit-collections data observed by the authors (see Sec. 5): while account covariates such as FICO score show significant variation across accounts/holders, they remain essentially constant for any given account over a typical collection cycle of four months. The fixed-effect covariate vector  $y$  can be adjusted periodically, resulting in a moving-horizon approach.

The account-specific information vector  $y$  determines, in large part, the account quality in terms of its collectability (i.e., repayment likelihood). Specifically, the intensity parameters  $\kappa$ ,  $\delta_1$ ,  $\delta_2$  depend on  $y$ . The parameter  $\kappa$  determines how fast the intensity approaches its long-run steady-state (in the absence of further jumps), effectively determining the memory of the system. The parameter  $\delta_1 = [\delta_{11}, \delta_{12}]^\top$  determines how past repayment events affect future repayment events. Its components determine the importance of the account holder’s willingness-to-repay (via  $\delta_{11}$ ) and her ability-to-repay (via  $\delta_{12}$ ), respectively. Lastly, the parameter  $\delta_2$  determines the sensitivity of the intensity with respect to account-treatment actions. The latter include bank-level actions such as moving the account from one collection phase (and agency) to the next (agency), adjusting commissions paid to collection agencies, as well as extending settlement offers, and agency-level interventions such as sending form letters, establishing first-party contacts, offering repayment plans, and filing lawsuits. The treatment schedule is modelled by a right-continuous, piecewise constant, vector-valued deterministic jump process  $a_t$  of dimension  $n_a$ , each component of which describes one type of account treatment (e.g., payment-plan offers or form letters). The pre-formulated treatment schedule  $a_t$  generates, via the SDE (2), jumps in the repayment-intensity process  $\lambda_t$  whenever an action is carried out. Because of the exponential decay, the effect of any given treatment intervention diminishes over time.

### 2.3.2 Group Covariates

Groups of similar accounts are influenced by time-varying group covariates. For example, macroeconomic covariates, such as inflation rate, unemployment rate, CPI, GDP, and prime rate, contain relevant aggregate information about the economy as a whole. The vector-valued group-covariate process  $X_t$  determines the long-run intensity  $\lambda_\infty(X_t)$  in Eq. (2) for an account group. To model the dynamics of these covariates we assume that  $X_t$  uniquely solves the SDE

$$dX_t = \mu(X_t) dt + \sigma(X_t) dW_t, \quad (3)$$

where  $W_t$  is a standard Wiener process in  $\mathbb{R}^{n_x}$ ;  $\mu$  and  $\sigma\sigma^\top$  are affine functions mapping  $\mathbb{R}^{n_x}$  to  $\mathbb{R}^{n_x}$  and  $\mathbb{R}^{n_x \times n_x}$ , respectively.<sup>11</sup> One can interpret the SDE (3) as the limit of an autoregressive conditional heteroskedasticity (ARCH) model, when the length of the time step in the underlying stochastic difference equation is taken to zero (Nelson 1990).

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<sup>11</sup>Duffie and Kan (1996) obtain sufficient conditions on  $\mu$  and  $\sigma$  which guarantee the existence and uniqueness of  $X_t$  in the domain of admissible covariates,  $\mathcal{D} = \{x \in \mathbb{R}^{n_x} : \sigma(x)\sigma^\top(x) \geq 0\}$ , where the SDE (3) is nondegenerate. For any such  $\mu$  and  $\sigma$ , Prop. A1 in Appendix A provides necessary and sufficient conditions that ensure  $\lambda_t$  is well defined.

The ARCH-process family has been widely applied in the econometrics literature to model the dynamics of financial time series (Bollerslev et al. 1992). Our modeling choice here is motivated on the one hand by the high heteroskedasticity in our observations, which is consistent with numerous empirical studies on financial data (Bollerslev et al. 1994), and on the other hand by the aforementioned asymptotic connection between ARCH and diffusion models.

### 3 Performance Quantification

#### 3.1 Repayment Probability

Given a positive time horizon  $\tau$  and the current time  $t$ , up to which information is available, we seek to determine the probability that over the interval  $(t, t + \tau]$  the total repayment of an account with current outstanding balance  $B_t > 0$  exceeds a given threshold  $B \in [0, B_t)$ . This amounts to computing the  $\tau$ -horizon repayment probability

$$\Pi_{t,t+\tau}(B/B_t) \triangleq P\left\{ \sum_{i=\mathcal{N}_t+1}^{\mathcal{N}_{t+\tau}} Z_i > B \mid \mathcal{F}_t \right\}, \quad (4)$$

where  $\mathcal{F}_t$  contains all currently available information, of which—as shown next—only the ratio  $B/B_t$  matters. The repayment probability in Eq. (4) constitutes our dynamic collectability score (DCS), an explicit expression of which can be obtained based on the dynamic repayment model in Eqs. (2) and (3). When passing from absolute to relative repayments as in Eq. (1), the right-hand side of Eq. (4) becomes

$$\begin{aligned} P\left\{ B_t - \sum_{i=\mathcal{N}_t+1}^{\mathcal{N}_{t+\tau}} Z_i < B_t - B \mid \mathcal{F}_t \right\} &= P\left\{ B_t \prod_{i=\mathcal{N}_t+1}^{\mathcal{N}_{t+\tau}} (1 - R_i) < B_t - B \mid \mathcal{F}_t \right\} \\ &= P\left\{ \sum_{i=\mathcal{N}_t+1}^{\mathcal{N}_{t+\tau}} Q_i < \log\left(\frac{B_t - B}{B_t}\right) \mid \mathcal{F}_t \right\}, \end{aligned}$$

where  $Q_i \triangleq \log(1 - R_i)$  denotes the logarithm of the outstanding balance percentage after the  $i$ -th relative repayment. Provided the conditional characteristic function (CCF) of the corresponding cumulative logarithmic relative-repayment process  $\mathcal{Q}_{t+\tau} - \mathcal{Q}_t \triangleq \sum_{i=\mathcal{N}_t+1}^{\mathcal{N}_{t+\tau}} Q_i$  is known, Lévy's inversion theorem can be used to compute this  $\tau$ -horizon repayment probability. Next, the repayment CCF is derived explicitly to compute the DCS.

#### 3.2 Collectability Scoring

Defining  $\hat{X}_t = [\mathcal{Q}_t, \lambda_t, X_t^\top]^\top$ , Eqs. (2) and (3) together specify a combined affine jump-diffusion model in which one of the jump factors is deterministic. Specifically,  $\hat{X}_t$  solves

$$d\hat{X}_t = \hat{\mu}(\hat{X}_t)dt + \hat{\sigma}(\hat{X}_t)d\hat{W}_t + \hat{\delta}_1 d\hat{J}_t + \hat{\delta}_2 da_t, \quad (5)$$

where  $\hat{W}_t$  is a standard Wiener process in  $\mathbb{R}^{n_{\hat{x}}}$ , and all the remaining model parameters and variables are suitably augmented.<sup>12</sup>

The conditional characteristic function of  $\mathcal{Q}_{t+\tau} - \mathcal{Q}_t$  is a special form of the transform that appears naturally in affine term-structure models of interest rates, e.g., to price zero-coupon bonds and European options (Duffie et al. 2000) or to value defaultable bonds and sovereign debt (Gibson and Sundaresan 1999; Merrick 1999). The key difference here is the presence of deterministic jumps to incorporate the bank's collection strategy and thus to optimize the account value by appropriately varying this strategy. The following result, formulated in a more general setting in Appendix A, extends a statement by Duffie et al. (2000, Prop. 1) to affine jump-diffusion processes with deterministic jump terms.

**Proposition 1** (Repayment CCF). *Let  $\theta_m$ , for  $m \in \{\underline{m}, \dots, \bar{m}\}$ , denote the known jump arrival times of  $a_s$ , for  $s \in (t, t + \tau]$ , and let  $(\hat{\alpha}, \hat{\beta})$  be the (unique) solution to the initial-value problem*

$$\dot{\hat{\alpha}}_s + \hat{\mu}_0^\top \hat{\beta}_s + \hat{\beta}_s^\top \hat{\sigma}_0 \hat{\sigma}_0^\top \hat{\beta}_s / 2 = 0, \quad \hat{\alpha}_{t+\tau} = 0, \quad (6)$$

$$\dot{\hat{\beta}}_s + \hat{\mu}_1^\top \hat{\beta}_s + \hat{\beta}_s^\top \hat{\sigma}_1 \hat{\sigma}_1^\top \hat{\beta}_s / 2 = \left(1 - \mathbb{E}[e^{\hat{\beta}_s^\top \hat{\delta}_1 \Delta \hat{J}}]\right) \hat{e}_2, \quad \hat{\beta}_{t+\tau} = \zeta \hat{e}_1, \quad (7)$$

on  $[t, t + \tau]$ , where  $\hat{e}_l \in \mathbb{R}^{n_{\hat{x}}}$  is the  $l$ -th Euclidean basis vector, and  $\zeta \in \mathbb{C}$  is given.<sup>13</sup> Under a set of technical conditions, provided in Appendix A, the repayment CCF is given by

$$\mathbb{E}[e^{\zeta(\mathcal{Q}_{t+\tau} - \mathcal{Q}_t)} | \mathcal{F}_t] = \exp \left[ -\Delta \hat{\alpha}_{t,t+\tau} - \Delta \hat{\beta}_{t,t+\tau}^\top \hat{X}_t + \sum_{m=\underline{m}}^{\bar{m}} \hat{\beta}_{\theta_m}^\top \hat{\delta}_2 \Delta a_{\theta_m} \right], \quad (8)$$

where  $\Delta a_{\theta_m} \triangleq a_{\theta_m} - a_{\theta_m^-}$ ,  $\Delta \hat{\alpha}_{t,t+\tau} \triangleq \hat{\alpha}_{t+\tau} - \hat{\alpha}_t$ , and  $\Delta \hat{\beta}_{t,t+\tau} \triangleq \hat{\beta}_{t+\tau} - \hat{\beta}_t$ .

Similar to Duffie et al. (2000), the above result is obtained by noticing that  $\mathbb{E}[e^{\zeta(\mathcal{Q}_{t+\tau} - \mathcal{Q}_t)} | \mathcal{F}_s]$  is martingale for  $s \in [t, t + \tau]$ ; hence its drift has to be zero. This together with the affine structure of the SDE (5) provides us with ODEs (6)–(7). The ODE (7) is a generalized Riccati equation, where the term  $\hat{\beta}_s^\top \hat{\sigma}_1 \hat{\sigma}_1^\top \hat{\beta}_s$  is a shorthand description of a vector in  $\mathbb{R}^{n_{\hat{x}}}$ , such that  $\left[\hat{\beta}_s^\top \hat{\sigma}_1 \hat{\sigma}_1^\top \hat{\beta}_s\right]_l \equiv \hat{\beta}_s^\top \hat{\sigma}_1^l (\hat{\sigma}_1^l)^\top \hat{\beta}_s$  for  $l \in \{1, \dots, n_{\hat{x}}\}$ . For special cases (e.g., when  $n_{\hat{x}} = 1$  and  $\hat{\delta}_1 = 0$ ) an analytical solution to (6)–(7) can be obtained (Jódar and Navarro 1991); numerical solutions are available using the various standard integration methods for ODEs (Butcher 2008).

**Proposition 2** (Dynamic Collectability Score). *The  $\tau$ -horizon repayment probability in Eq. (4) is equal to*

$$\Pi_{t,t+\tau}(B/B_t) = \frac{1}{2} - \int_{-\infty}^{\infty} \frac{e^{-\Delta \hat{\alpha}_{t,t+\tau}(jw) - \Delta \hat{\beta}_{t,t+\tau}^\top(jw) \hat{X}_t + \sum_{m=\underline{m}}^{\bar{m}} \hat{\beta}_{\theta_m}^\top(jw) \hat{\delta}_2 \Delta a_{\theta_m} - jw \log(1 - \frac{B}{B_t})}}{2\pi(jw)} dw, \quad (9)$$

<sup>12</sup> $\hat{\mu}(\hat{x}) = \hat{\mu}_0 + \hat{\mu}_1 \hat{x}$  in which  $(\hat{\mu}_0, \hat{\mu}_1) \in \mathbb{R}^{n_{\hat{x}}} \times \mathbb{R}^{n_{\hat{x}} \times n_{\hat{x}}}$  combines the drift terms of Eqs. (2) and (3); and  $\hat{\sigma}(\hat{x}) \hat{\sigma}^\top(\hat{x}) = \hat{\sigma}_0 + \sum_{l=1}^{n_{\hat{x}}} \hat{\sigma}_1^l \hat{x}_l$  where  $(\hat{\sigma}_0, (\hat{\sigma}_1^l)_{l=1}^{n_{\hat{x}}}) \in \mathbb{R}^{n_{\hat{x}} \times n_{\hat{x}}} \times \mathbb{R}^{n_{\hat{x}} \times n_{\hat{x}} \times n_{\hat{x}}}$  expands  $\sigma(x) \sigma^\top(x)$  to a function from  $\mathbb{R}^{n_{\hat{x}}}$  to  $\mathbb{R}^{n_{\hat{x}} \times n_{\hat{x}}}$ . Furthermore,  $\hat{J}_t = [\mathcal{Q}_t, \mathcal{N}_t, \mathcal{R}_t]$  has i.i.d. jumps denoted by  $\Delta \hat{J}$ , and  $(\hat{\delta}_1, \hat{\delta}_2) \in \mathbb{R}^{n_{\hat{x}} \times 3} \times \mathbb{R}^{n_{\hat{x}} \times n_a}$  is obtained by properly resizing  $(\delta_1, \delta_2)$  (see Eqs. (25)–(27) in Appendix A).

<sup>13</sup>The solution  $(\hat{\alpha}_s, \hat{\beta}_s)$  depends on the given boundary value  $(0, \zeta)$  for  $s = t + \tau$ . The value  $\zeta$  is in practice either complex or real,  $\zeta \in \{jw, 1\}$ , where  $j = \sqrt{-1}$  and  $w \in \mathbb{R}$ . To keep notation simple the parametric dependence is suppressed when not needed explicitly.

where  $(\hat{\alpha}, \hat{\beta})$  solves the initial-value problem (6)–(7) for  $\zeta = jw$ .

**Example 1.** Consider the case where  $\hat{\delta}_1 = 0$ , and  $\lambda_\infty$  is constant, i.e., no time-varying macroeconomic indicators are present. In this setting, the repayment timing process  $(T_i, i \geq 1)$  is independent of the relative-repayment process  $(R_i, i \geq 1)$ . The intensity  $\lambda_s$ , for  $s \in [t, t + \tau]$ , is then

$$\lambda_s = \lambda_\infty + (\lambda_t - \lambda_\infty)e^{-\kappa(s-t)} + \sum_{\theta_m \in (t, s]} \delta_2^\top \Delta a_{\theta_m} e^{-\kappa(s-\theta_m)},$$

where the current intensity  $\lambda_t$  is known. Under the above assumptions, the inverse Fourier transform in Eq. (9) can be obtained indirectly by calculating the probability mass function of  $\mathcal{N}_{t+\tau} - \mathcal{N}_t$  via evaluating the derivatives of

$$\mathbb{E}[\zeta^{(\mathcal{N}_{t+\tau} - \mathcal{N}_t)} | \mathcal{F}_t] = \exp \left[ -\Delta \hat{\alpha}_{t, t+\tau} - \Delta \hat{\beta}_{t, t+\tau}^\top \hat{X}_t + \sum_{m=m}^{\bar{m}} \hat{\beta}_{\theta_m}^\top \hat{\delta}_2 \Delta a_{\theta_m} \right], \quad (10)$$

at  $\zeta = 0$ , where  $\hat{X}_t = [\mathcal{N}_t, \lambda_t]^\top$ , and  $(\hat{\alpha}, \hat{\beta})$  solves the ODE in Prop. 1 with the boundary condition  $(\hat{\alpha}_{t+\tau}, \hat{\beta}_{t+\tau}) = (0, \log(\zeta) \hat{e}_1)$ . Thus,

$$\hat{\alpha}_s = \frac{\zeta - 1}{\kappa} \lambda_\infty \left( e^{-\kappa(t+\tau-s)} - 1 + \kappa(t + \tau - s) \right),$$

$$\hat{\beta}_s = \left[ \begin{array}{c} \log(\zeta) \\ \frac{\zeta - 1}{\kappa} (1 - e^{-\kappa(t+\tau-s)}) \end{array} \right],$$

for all  $s \in [t, t + \tau]$ . The argument of the exponential function in Eq. (10) is therefore equal to the area under  $\lambda_s$  when  $s \in [t, t + \tau]$ , where  $\hat{\alpha}_t$  represents the area generated by the fixed part of  $\lambda$  (i.e., long-run steady-state  $\lambda_\infty$ ) while  $\hat{\beta}_t$  and  $\hat{\beta}_{\theta_m}$  represent the area contributed by the transient parts (i.e., the intensity's current value  $\lambda_t$  and the jumps  $\hat{\delta}_2 \Delta a_{\theta_m}$ ). Denoting the cumulative distribution function (CDF) of  $Q_i$  by  $F_Q$ , we have

$$\Pi_{t, t+\tau}(B/B_t) = \sum_{n=1}^{\infty} P\{\mathcal{N}_{t+\tau} - \mathcal{N}_t = n\} F_Q^{*n} \left( \log \left( 1 - \frac{B}{B_t} \right) \right),$$

in which the counting process follows a Poisson distribution,

$$\mathcal{N}_{t+\tau} - \mathcal{N}_t \sim \text{Pois} \left( \lambda_\infty \tau + \frac{1 - e^{-\kappa\tau}}{\kappa} (\lambda_t - \lambda_\infty) + \sum_{m=m}^{\bar{m}} \frac{1 - e^{-\kappa(t+\tau-\theta_m)}}{\kappa} \delta_2^\top \Delta a_{\theta_m} \right),$$

and  $F_Q^{*n}$  is the  $n$ -fold convolution of  $F_Q$ . In the general setting,  $\hat{\alpha}_t$  and  $\hat{\beta}_t$  can be interpreted in the same way.<sup>14</sup>  $\square$

<sup>14</sup>The method used in this example for calculating the inverse Fourier transform of Eq. (9) can be extended to the general setting by discretizing the  $Q_i$ . This approach is used in Sec. 5.2.2 to avoid numerical complications (Gibbs effect) that may arise when computing the DCS.

### 3.3 Account Valuation

Given a description of the repayment process based on information up to the present ( $\mathcal{F}_t$ ) and the action process  $a$ , with scheduled interventions  $\Delta a_{\theta_m}$  at times  $\theta_m \in (t, t + \tau]$ , we are now interested in determining the expected  $\tau$ -horizon present value of an account, denoted by  $V_{t,t+\tau}(a)$ .

**Proposition 3** (Account Valuation). *Let  $\rho$  be a discount factor. The expected  $\tau$ -horizon present value of an account with outstanding balance  $B_t$  is*

$$V_{t,t+\tau}(a) = B_t \int_{(t,t+\tau]} e^{-\rho(s-t)} d\nu_t(s|a); \quad (11)$$

the cash-flow measure  $\nu_t(s|a)$  represents the expected cumulative repayment over  $(t, t + s]$ :

$$\nu_t(s|a) = 1 - e^{-\Delta \hat{\alpha}_{t+\tau-s,t+\tau} - \Delta \hat{\beta}_{t+\tau-s,t+\tau}^\top \hat{X}_t + \sum_{\theta_m \in (t,t+s]} \hat{\beta}_{\tau-s+\theta_m}^\top \hat{\delta}_2 \Delta a_{\theta_m}}, \quad (12)$$

where  $(\hat{\alpha}, \hat{\beta})$  is the solution to the initial-value problem (6)–(7) for  $\zeta = 1$ .

**Example 2.** Under the assumptions of Ex. 1, the cash-flow measure of Eq. (12) becomes

$$\nu_t(s|a) = 1 - e^{-\bar{r}\lambda_\infty s - \frac{\bar{r}}{\kappa}(1-e^{-\kappa s})(\lambda_t - \lambda_\infty) - \sum_{\theta_m \in (t,t+s]} \frac{\bar{r}}{\kappa}(1-e^{-\kappa(t+s-\theta_m)})\delta_2^\top \Delta a_{\theta_m}},$$

where  $\bar{r} \triangleq \mathbb{E}[R_i]$  is the expected relative repayment. For any given discount factor  $\rho$ , one can then obtain the account value by calculating the integral in Eq. (11) numerically. From an optimization perspective, it is interesting that taking a more forceful action at the scheduled time, or alternatively, taking a given action earlier, leads to a first-order stochastically dominant shift of the cumulative repayment distribution, resulting in a larger cash-flow measure (Kwieciński and Szekli 1991). The corresponding expected benefits in terms of account value may be offset by the higher cost of a larger action as well as the time value of money of the capital expenditure required to finance the additional or earlier collection effort. Chehrazi et al. (2014) investigate this tradeoff in a general infinite-horizon framework.  $\square$

## 4 Model Identification

Consider now the problem of identifying the model parameters for a group of  $\bar{k}$  accounts. For each account  $k$  in the portfolio  $\mathcal{K} = \{1, \dots, \bar{k}\}$ , account-specific data is available over the study interval  $[0, h^k]$ , with horizon  $h^k \geq \tau$ . The data includes the repayment history  $N^k = (T_i^k, R_i^k, i \geq 1)$  and a process  $a^k$  of scheduled account-treatment actions. Relevant for all accounts in the portfolio  $\mathcal{K}$  is the group-covariate process  $X$  whose path is also observed during the study interval  $[0, h^k]$ ,  $k \in \mathcal{K}$ .

The tuple of model parameters,  $\vartheta = [\vartheta_\lambda; \vartheta_X]$ , has separate entries for the intensity process ( $\vartheta_\lambda$ ) and for the group-covariate process ( $\vartheta_X$ ). Since the dynamics of the group-covariate process do not depend on  $\vartheta_\lambda$ , the group-covariate-process parameter vector  $\vartheta_X$  can be estimated based on the observed sample path for  $X$  and independently of the observed repayment process  $N^k$  and the parameter vector  $\vartheta_\lambda$ . By contrast, the estimation of  $\vartheta_\lambda$  depends on  $\vartheta_X$

(for a given realization of  $X$ ); however, this dependency is only through the constraints that ensure positivity of the intensity process (see Prop. A1). These constraints, together with a goodness-of-fit condition, bound the domain  $\Theta_\lambda$  of feasible  $\vartheta_\lambda$  but otherwise do not affect the estimation objective (the likelihood function or a GMM-loss). Hence, a sequential estimation procedure, identifying first  $\vartheta_X$  and then  $\vartheta_\lambda$ , while computationally advantageous is no less efficient than an analytically more complex joint estimation of  $\vartheta$ .

## 4.1 Estimation of the Group-Covariate Process

The estimation of an affine diffusion process  $X_t$  with parameter tuple  $\vartheta_X = [\mu, \sigma\sigma^\top]$  has been discussed extensively in the literature on dynamic asset pricing (Singleton 2001). The methods considered there are based on either maximum-likelihood estimation (MLE) or the generalized method of moments (GMM). They are usually developed for specific financial applications, given a sample of  $X_t$  (or dependent variables such as prices and/or yields). While in our application, both MLE and GMM can be used to obtain an estimate  $\tilde{\vartheta}_X^*$  of  $\vartheta_X$ , GMM is often preferred, since knowledge about the CCF for affine (jump-)diffusion processes (see Prop. A2) can be used to develop computationally tractable and asymptotically efficient estimators.<sup>15</sup>

## 4.2 Estimation of the Repayment Process

Conditional on the estimate  $\tilde{\vartheta}_X^*$  for the model (3) of the group-covariate process, the identification of the repayment process  $N = (T_i, R_i, i \geq 1)$  can be accomplished in two steps: in the first step, we estimate the distribution  $F_R$  for the relative-repayment process  $(R_i, i \geq 1)$ ; in the second step, we estimate  $\vartheta_\lambda$  by fitting the intensity process  $\lambda_t$  to the observed repayment-timing process  $(T_i, i \geq 1)$ . The two steps of the procedure are detailed in turn.

**Step 1:** As an estimate of  $F_R$  we use the empirical CDF

$$\tilde{F}_R(r) = \frac{\sum_{k=1}^{\bar{k}} \sum_{i=1}^{\mathcal{N}_{h^k}^k} \mathbf{1}_{\{R_i^k \leq r\}}}{\sum_{k=1}^{\bar{k}} \mathcal{N}_{h^k}^k},$$

given the data  $(R_i^k, 1 \leq i \leq \mathcal{N}_{h^k}^k)_{k=1}^{\bar{k}}$ , taking into account Eq. (1). The i.i.d. assumption (or rather modeling choice) in Sec. 2.1 implies the consistency and unbiasedness of this nonparametric estimate.

**Step 2:** The parameter  $\vartheta_\lambda = [\lambda_\infty, \kappa, \delta_1, \delta_2]$  is estimated using two alternative methods: MLE and GMM; the latter is implemented via prediction-error minimization (PEM). The operational purpose of the repayment-process model determines which method is preferable. While MLE is asymptotically efficient, using the model parameters that maximize the likelihood of the observed data will not necessarily lead to a decision that minimizes the loss in business terms. In the presence of a modeling error,<sup>16</sup> GMM often provides better out-of-

<sup>15</sup>The stationarity and ergodicity of the affine (jump-)diffusion processes required for establishing consistency and asymptotic behavior of MLE and GMM estimators are discussed by Glasserman and Kim (2010) and Zhang and Glynn (2012).

<sup>16</sup>Modeling errors cannot be eliminated in practice, since for any finite amount of data there exist different models, each of which cannot be rejected—at a given level of statistical significance.

sample results, since appropriate moment conditions can be used to construct loss functions that are closely related to business objectives. This observation is the underlying idea of recent investigations in operational statistics and robust estimation where model estimation and performance optimization are combined (Besbes et al. 2010; Chehrizi and Weber 2010).

*Maximum-Likelihood Estimation (MLE).* The joint probability density function for the arrival times  $(T_i, 1 \leq i \leq \mathcal{N}_t)$  of the repayment process  $N$  can be derived by establishing a link between the conditional density function of the interarrival times and the intensity process (Rubin 1972).<sup>17</sup> This leads to a compact expression for the log-likelihood function of  $(T_i^k, 1 \leq i \leq \mathcal{N}_{h^k}^k)$ ,

$$L((T_i^k, 1 \leq i \leq \mathcal{N}_{h^k}^k) | (X_s, s \in [0, h^k]), (R_i^k, 1 \leq i \leq \mathcal{N}_{h^k}^k), \tilde{\vartheta}_\lambda) = - \int_0^{h^k} \lambda_s^k(\tilde{\vartheta}_\lambda) ds + \int_0^{h^k} \ln(\lambda_{s^-}^k(\tilde{\vartheta}_\lambda)) d\mathcal{N}_s^k, \quad (13)$$

where the estimated intensity process  $\lambda_s^k = \lambda_s^k(\tilde{\vartheta}_\lambda)$ , in analogy to Eq. (2), is given by

$$d\lambda_s^k = \tilde{\kappa}(y^k)(\tilde{\lambda}_\infty(X_s) - \lambda_s^k) ds + \tilde{\delta}_1^\top(y^k) dJ_t^k + \tilde{\delta}_2^\top(y^k) da_t^k, \quad s \in \mathbb{R}_+,$$

with corresponding parameter estimate  $\tilde{\vartheta}_\lambda = [\tilde{\lambda}_\infty, \tilde{\kappa}, \tilde{\delta}_1, \tilde{\delta}_2]$ , for any account  $k \in \mathcal{K}$ . The maximum-likelihood estimator  $\tilde{\vartheta}_\lambda^*$  is then the  $\tilde{\vartheta}_\lambda$  that maximizes Eq. (13).

*Generalized Method of Moments (GMM).* As an alternative to MLE, we consider an estimator that minimizes a loss function that is in accordance with a business-relevant measure such as the expected present value of an account. For any given prediction horizon  $\tau$ , time  $t \leq h^k - \tau$ , and parameter estimate  $\tilde{\vartheta}_\lambda$ , Prop. 3 yields the expected present value of any individual account  $k$ :  $V_{t,t+\tau}^k(a^k | \tilde{\vartheta}_\lambda) = \mathbb{E}[\sum_i e^{-\rho T_i^k} Z_i^k \mathbf{1}_{\{t < T_i^k \leq t+\tau\}} | \mathcal{F}_t, a^k, \tilde{\vartheta}_\lambda]$ . The model can then be identified by choosing an estimate  $\tilde{\vartheta}_{\lambda,t}^*$  that minimizes the square deviation between model prediction and observed data, so

$$\tilde{\vartheta}_{\lambda,t}^* \in \arg \min_{\tilde{\vartheta}_\lambda \in \Theta_\lambda} \left\{ \sum_{k \in \mathcal{K}} (V_{t,t+\tau}^k(a^k | \tilde{\vartheta}_\lambda) - v_{t,t+\tau}^k)^2 \right\}, \quad (14)$$

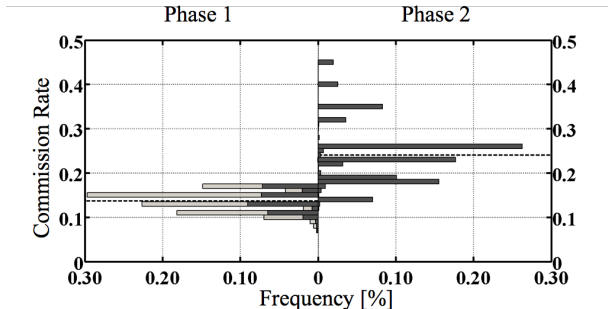
where  $v_{t,t+\tau}^k = \sum_i e^{-\rho T_i^k} Z_i^k \mathbf{1}_{\{t < T_i^k \leq t+\tau\}}$  is the observed  $\tau$ -horizon value of the  $k$ -th account. This GMM criterion is appropriate when valuation, either at the account level or the portfolio level, is of interest (for example when pricing uncollectable accounts, so-called “dead-reserves,” for debt leasing). As detailed in Sec. 5, MLE and GMM provide similar structural insights about the repayment process, and they both exhibit a similar out-of-sample scoring performance. However, the valuation error (see Eq. (15)) of the GMM estimator is considerably smaller (by about 50 percent) than that of MLE (see Sec. 5.2.1). In applications with limited data, such performance improvements—as well as the built-in robustness with respect to modeling error—often outweigh the loss in asymptotic efficiency, thus rendering GMM-based estimation a viable alternative to MLE.

**REMARK 2.** Both MLE and GMM require observing  $X_t$  over the entire study interval. In practice, some group covariates may only be observed at discrete time instances. In such a

<sup>17</sup>The likelihood function can also be obtained by a change of measure (Giesecke and Schwenkler 2012).

Attributes		Min	Max	Avg
$y^k$	Placement balance [\$]	500	1500	945
	FICO score	551	700	608
$h^k$	Days in collections	8	550	208
$\mathcal{N}^k$	Payment count	0	16	1.86
$c_1^k$	Phase 1 commission rate [%]	7	23	13.6
$\theta_2^k$	Transition to the 2nd collection phase [day]	30	515	151
$c_2^k$	Phase 2 commission rate [%]	8	45	23.5

(a)



(b)

Figure 1: Description of the dataset: (a) Summary statistics; (b) Distribution of commission rates.

case, one can construct the relevant continuous sample path by interpolating between the observed values. Alternatively, one can work with an intensity process  $\Lambda_t$  specified under the internal history of  $N$  by taking the expectation of  $\lambda_t$  with respect to  $X$ , thus removing the dependence of  $\lambda_t$  on the sample path of  $X$  (Brémaud 1981, Thm. 14). This expectation can be evaluated explicitly by adapting Prop. A2.<sup>18</sup>

## 5 Applications

We now employ the framework developed thus far to estimate the collectability and the value of delinquent credit-card accounts. For this, we use a large dataset of account-specific credit-collections data and identify our model using both MLE and GMM estimation methods, as detailed in Sec. 5.1. We then compare the two methods and highlight the operational gains that can be obtained from the latter when adapting it to business-relevant objectives (Sec. 5.2.1). The DCS constructed in Prop. 2 is used to determine the payback probability and value accounts (Sec. 5.2.2). The out-of-sample prediction performance of DCS at different instances of the collections process is illustrated by Receiver Operating Characteristic (ROC) curves, and compared to a bank-internal scoring (BIS) system (Sec. 5.2.3). Finally, the model is used to optimize commission rates for debt-leasing agencies in a typical two-phase external collections process (Sec. 5.2.4).

### 5.1 Dataset and Model

The available dataset contains approximately 6,600 delinquent accounts with a total outstanding balance of \$6 million, placed in collections between 2004 and 2006; see Fig. 1a for summary statistics. To evaluate the prediction performance of the model we limit account-specific covariates  $y$  to two attributes: outstanding balance at placement and pre-default

<sup>18</sup>Specifically,  $\Lambda_t$  solves  $d\Lambda_t = \kappa(y)(\lambda_\infty(\bar{x}_t) - \Lambda_t) dt + \delta_1^\top(y) dJ_t + \delta_2^\top(y) da_t$ , where  $\bar{x}_t \triangleq \mathbb{E}[X_t | \mathcal{F}_0]$  solves  $d\bar{x}_t = \mu(\bar{x}_t) dt$ .



FICO score.<sup>19</sup> We consider two high-level collection actions: assigning an account to a new collection agency (effectively starting a new collection phase) and setting commission rates. In our dataset, an account’s repayment behavior is tracked over two collection phases. In order to avoid conflict of interest, e.g., agencies’ artificially delaying collections, a commission rate is increased only when an account is reassigned to a different agency. Hence, the process of scheduled account-treatment actions  $a$  includes at most only two interventions ( $\bar{m} = 2$ ):  $\Delta a_{\theta_1} = [1, c_1]^\top$  where  $\theta_1 = 0$  is the placement time for phase 1; and  $\Delta a_{\theta_2} = [1, c_2 - c_1]^\top$  where  $\theta_2$  denotes the placement time for phase 2 (if applicable). The distributions of account-specific commission rates  $c_1$  and  $c_2$  for phase 1 and 2, respectively, are summarized in Fig. 1b; some accounts were transitioned to the second phase (shaded dark), on average after 120 days of nonpayment in the first phase. The dashed lines illustrate the average commission rates for the two collection phases.

In addition to the account-specific covariates discussed earlier, our dataset includes a BIS derived from quasi-static regression models (e.g., Crook et al. 2007) and heuristic data-mining methods; it provides a natural benchmark. Since the BIS is often precomputed at the outset for later use, we enhance its dynamic performance using Bayesian updating, termed ‘updated BIS’ (uBIS), based on the observed repayment history for each account.

**Repayment-Process Discretization.** The repayment process  $(T_i^k, Z_i^k, 1 \leq i \leq \mathcal{N}_{h^k}^k)$  of account  $k$  is approximated by lump-sum payments  $Z_i^k$  at the collection-report times  $T_i^k$ , where  $Z_i^k$  corresponds to the sum of all repayments between  $T_{i-1}^k$  and  $T_i^k$ . Fig. A1 shows the empirical CDF of the relative repayments,  $\tilde{F}_R$ . Pearson’s chi-squared test rejects the hypothesis that the  $R_i^k$  are independent at the account level (i.e., for a fixed  $k$ ). In the available dataset, the average number of payments is small (approximately 2). Hence, the hypothesis that the  $R_i^k$  are identically distributed cannot be tested at the account level. At the aggregate level, when pooling the relative-repayment sequences for all  $k \in \mathcal{K}$ , the Empirical Generalized Runs test does not reject the hypothesis that the  $R_i^k$  are i.i.d.<sup>20</sup>

**Account Stratification.** The dataset can be split into random “training” and “testing” subgroups by stratifying either across time or across accounts. In the former version, the model is trained over the initial portion (e.g., 60 percent) of the study interval and the out-of-sample performance is determined over the remaining length. This mode of analysis is attractive when a decision maker is interested in the optimal account-specific timing and design of a settlement offer. In the latter version, the dataset is divided into training and testing subgroups across accounts, so parameters can be identified over the full length of the collections process. This is useful when a decision maker is interested in valuing a portfolio of accounts (at the aggregate level, including overlapping generations). The latter version is adopted here for account-valuation and collection-action optimization. Thus, 60% of accounts (3,757) were randomly selected for training and the remaining 40% of accounts (2,504) for testing. In these two subgroups, at least one payment was made by 83% (3,115) and 84% (2,096) of all accounts, respectively.<sup>21</sup>

<sup>19</sup>Bank-internal datasets usually contain additional attributes such as credit limit, interest rate, utilization rate, and past-due/personal-bankruptcy events. In our implementation, both the placement balance and the pre-default FICO score are linearly mapped to  $[0, 1]$ .

<sup>20</sup>The corresponding values of the EGR test statistics  $\tilde{T}_{1,n}(\mathbb{S}_1)$ ,  $\tilde{T}_{1,n}(\mathbb{S}_2)$ ,  $\tilde{T}_{\infty,n}(\mathbb{S}_1)$ , and  $\tilde{T}_{\infty,n}(\mathbb{S}_2)$  as defined in Cho and White (2011) are 0.0064, 0.0023, 0.597, and 0.1846, respectively, which are all majorized by the corresponding critical values at the 1%-level.

<sup>21</sup>The treatment of data censoring due to the fact that full repayment may be observed infrequently in typical repayment data is discussed in Appendix A.

$y$		$\tilde{\kappa}^*$	$\tilde{\delta}_1^*$		$\tilde{\delta}_2^*$		$\tilde{\lambda}_\infty^*$	
			$\tilde{\delta}_{11}^*$	$\tilde{\delta}_{12}^*$	$\tilde{\delta}_{21}^*$	$\tilde{\delta}_{22}^*$	$\tilde{\lambda}_{\infty,0}^*$	$\tilde{\lambda}_{\infty,1}^*$
1	( $\times 10^{-2}$ )	0.522 (0.066)	0.400 (0.126)	0.449 (0.300)	0 <sup>†</sup> (0.086)	0.778 (0.617)	0.288 (0.047)	-6.258 (1.264)
$B_0$	( $\times 10^{-2}$ )	0.291 (0.086)	1.356 (0.185)	-0.370 (0.470)	0 <sup>†</sup> (0.131)	-0.213 (0.985)	N/A	N/A
FICO	( $\times 10^{-2}$ )	0.913 (0.102)	1.311 (0.211)	-0.079 (0.514)	0 <sup>†</sup> (0.184)	12.584 (1.399)	N/A	N/A

Table 1: Parameter estimates for MLE (incl. standard error);  $0 \leq 0^\dagger \leq 10^{-3}$ .

$y$		$\tilde{\kappa}^*$	$\tilde{\delta}_1^*$		$\tilde{\delta}_2^*$		$\tilde{\lambda}_\infty^*$	
			$\tilde{\delta}_{11}^*$	$\tilde{\delta}_{12}^*$	$\tilde{\delta}_{21}^*$	$\tilde{\delta}_{22}^*$	$\tilde{\lambda}_{\infty,0}^*$	$\tilde{\lambda}_{\infty,1}^*$
1	( $\times 10^{-2}$ )	1.976 (9.032)	0.777 (14.607)	0.442 (24.441)	0 <sup>†</sup> (6.738)	0 <sup>†</sup> (13.246)	1.163 (2.111)	-25.266 (65.960)
$B_0$	( $\times 10^{-2}$ )	0.128 (13.058)	3.596 (15.440)	-0.442 (28.019)	0 <sup>†</sup> (2.552)	-0 <sup>†</sup> (8.487)	N/A	N/A
FICO	( $\times 10^{-2}$ )	0.262 (13.888)	-0.776 (12.945)	-0 <sup>†</sup> (12.183)	0 <sup>†</sup> (20.076)	13.363 (58.202)	N/A	N/A

Table 2: Parameter estimates for GMM (incl. standard error);  $0 \leq 0^\dagger \leq 10^{-3}$ .

**Parameter Estimation.** The average rate on one-month negotiable certificates of deposit (CD), taken from the Federal Reserve Bank’s H.15 periodic reports, serves as group-covariate process  $X_t$ . As noted in Sec. 5.1, the testing and training subgroups are tracked over the same study period: 2004 to 2006. To avoid the bias that would result from including study-period samples of  $X$  in the estimation of the intensity parameter  $\vartheta_\lambda$  (they should remain unknown for the testing subgroup), only observed covariates prior to  $t = 0$  (i.e., 2004) are taken for the estimation of  $\vartheta_X$ , while  $\Lambda_t$  (see Remark 2) is used to estimate  $\vartheta_\lambda$ .<sup>22</sup> Given  $\tilde{\vartheta}_X^*$ , the intensity parameter  $\vartheta_\lambda$  is estimated by both MLE and GMM methods. Table 1 and Table 2 summarize the results and also provide the (asymptotic) standard error (SE) for each estimator.<sup>23</sup> The validity of the model is confirmed by the Cramér-von Mises (CVM) goodness-of-fit test (see Appendix A for details).<sup>24</sup> The CVM-test rejects the model when  $\delta_1$  is forced to zero, which implies that the intensity process is indeed sensitive to the jump process  $J_t$ , thus justifying empirically the use of a self-exciting point process as a model for the stochastic repayments.

**Interpretation.** The first three components of  $\vartheta_\lambda = [\kappa, \delta_1, \delta_2, \lambda_\infty]$  are affine functions of the account-specific covariate vector  $y$ , while the last component is an affine function of the group covariate  $X_t$ . One can compare MLE and GMM estimates by examining the sign and magnitude of the coefficients for each affine function. The sign determines the directional dependence of the considered function with respect to the changes in the corresponding covariate. For example, the sign of  $\tilde{\lambda}_{\infty,1}^*$  is negative for both GMM and MLE. This suggests

<sup>22</sup>Based on 420 monthly samples for 1-month negotiable CDs, from 1970 to 2004, an estimate of  $(\mu_0, \mu_1)$  is obtained:  $(\tilde{\mu}_0^*, \tilde{\mu}_1^*) = (3.70 \times 10^{-5}, -6.089 \times 10^{-4})$  with the standard error of  $(3.75 \times 10^{-5}, 6.828 \times 10^{-4})$  calculated according to Hansen (1982).

<sup>23</sup>In our implementation  $\lambda_{0-} = \Lambda_{0-} = 0$ . This corresponds to the notion that if a delinquent account is not placed in collection, there will be no repayment. The SE of the MLE estimates are obtained by approximating the Fisher information (Berndt et al. 1974). The GMM estimates are derived by taking  $\rho = 0$ ,  $t = 0$ , and  $\tau = 120$  days (see Eq. (14)), with SE derived according to Hansen (1982).

<sup>24</sup>The values of the CVM-test for the MLE and GMM parameter estimates in the testing subgroup are 0.403 and 0.441, respectively. Both are majorized by the 5%-level critical value of 0.461.

that as the interest rate increases,  $\tilde{\lambda}_\infty^*$  decreases resulting in a smaller repayment probability and thus a lower account value. Similarly, the sign of the coefficient of  $\tilde{\kappa}^*$  for the FICO score is positive for both estimation methods. Thus, with increasing creditworthiness the inertia of the intensity process decreases, effectively shortening the memory of the repayment process. In other words, for a larger FICO score the past becomes less predictive of the future. Indeed, if a large FICO score was a good indicator for repayment behavior, the account should not have ended up in collections. The value of  $\tilde{\kappa}^*$  also increases with the outstanding balance  $B_0$  for both GMM and MLE estimates. This means that a higher placement balance also decreases the memory of the repayment intensity. When the balance is high, the current financial status becomes a better predictor than the account holder’s past financial status.

While the signs of MLE and GMM estimates tend to be consistent, their magnitudes may differ in part because coefficients can offset each other. Instead of directly comparing the latter, we therefore compare the values of the affine functions for a (fictitious) account with average covariate vector  $y_{\text{avg}}$  for the considered account segment (low pre-default balance, high FICO score). Table 3 shows that for both MLE and GMM there is a large difference in the sensitivities of the repayment-arrival intensity in Eq. (2) to the different components of the repayment process  $J_t = [\mathcal{N}_t, \mathcal{R}_t]^\top$ . Specifically,  $\tilde{\delta}_{11}^*(y_{\text{avg}}) \gg \tilde{\delta}_{12}^*(y_{\text{avg}})$ , so to predict repayment behavior in this segment the willingness-to-repay ( $\mathcal{N}_t$ ) proves significantly more important than ability-to-repay ( $\mathcal{R}_t$ ).<sup>25</sup> This suggests that testing actions with the potential to boost an account holder’s willingness-to-repay, such as filing a lawsuit or offering credit reinstatement upon full repayment, may constitute useful business experiments for this segment. For both MLE and GMM, the account-treatment sensitivities vary across actions ( $\tilde{\delta}_{21}^*(y_{\text{avg}}) \ll \tilde{\delta}_{22}^*(y_{\text{avg}})$ ): for the segment under consideration the mere placement of an account to a new agency proves less important than an increase in the commission rate. The repayment intensity  $\lambda_t$  in the SDE (2), and therefore also  $\Lambda_t$  (see Remark 2), is sensitive to a change of the commission rate; this sensitivity increases in the FICO score and diminishes with greater placement balance  $B_0$  (see the coefficients of  $\delta_{22}$  in Table 1 and Table 2).

## 5.2 Performance Analysis

In what follows, we compare the performance of the model for MLE against GMM. We then illustrate the computation of the dynamic collectability score as the backbone of the method. A comparison of the prediction performance for the two model-identification methods follows before we show how to use it for optimizing account treatment by setting commission rates.

### 5.2.1 MLE vs. GMM

The two identification methods yield similar intuition about repayment behavior in the segment  $\mathcal{K}$ . As shown in Sec. 5.2.3, their prediction performance is also similar. However, significant performance differences may arise for business-relevant objectives that are closely related to the PEM moment conditions. For example, when aggregating the expected values of different accounts in a segment, one obtains a business-relevant objective (“segment

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<sup>25</sup>Because the accounts have low balance and high pre-default FICO scores, the group is considered “highly collectable.” This may not hold for other segments, e.g., for low-FICO-score/high-balance accounts.

	$\tilde{\kappa}^*(y_{\text{avg}})$	$\tilde{\delta}_1^*(y_{\text{avg}})$		$\tilde{\delta}_2^*(y_{\text{avg}})$	
		$\tilde{\delta}_{11}^*(y_{\text{avg}})$	$\tilde{\delta}_{12}^*(y_{\text{avg}})$	$\tilde{\delta}_{21}^*(y_{\text{avg}})$	$\tilde{\delta}_{22}^*(y_{\text{avg}})$
MLE	0.010	0.015	0.003	0 <sup>†</sup>	0.056
GMM	0.021	0.021	0.002	0 <sup>†</sup>	0.052

(a)

		Relative Valuation Error
Training	MLE	11.8%
	GMM	5.9%
Testing	MLE	11.7%
	GMM	5.7%

(b)

Table 3: MLE vs. GMM: (a) Average model parameters; (b) Relative valuation error.

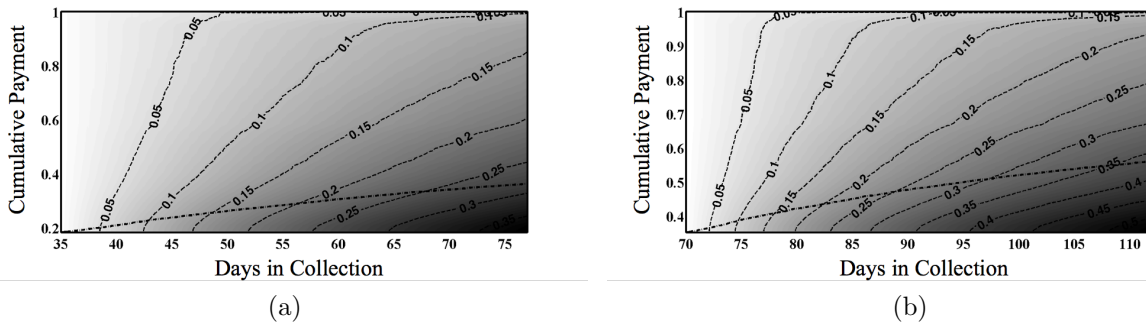
value”), for which the bank seeks accurate forecasts. The corresponding relative valuation error (RVE) is

$$\text{RVE} = \left| \frac{\sum_{k=1}^{\bar{k}} V_{t,t+\tau}^k(a^k) - \sum_{k=1}^{\bar{k}} v_{t,t+\tau}^k}{\sum_{k=1}^{\bar{k}} v_{t,t+\tau}^k} \right|. \quad (15)$$

As shown in Table 3b (for collection phase 1, with  $\rho = 0$ ,  $t = 0$ , and  $\tau = 120$  days) the RVE for the GMM method, implemented via PEM, clearly outperforms the RVE for MLE. The main reason for the performance difference is the presence of modeling errors (see footnote 16). As noted at the end of Sec. 5.1, the repayment-process model responds in a natural way to the various outside forces, such as repayment events or treatment actions. The linear intensity dynamics and the assumed i.i.d. relative repayment amounts should be considered a reasonable first-order approximation of reality.

### 5.2.2 Payback Probability

The DCS summarizes the probability of receiving an aggregate repayment in excess of a given threshold  $B$  over a given interval  $(t, t + \tau]$ . Fig. 2 depicts this repayment probability in shades of grey with corresponding iso-quantile lines for a sample account (in the testing subgroup).



(a)

(b)

Figure 2: Cumulative payback probability  $\Pi_{t,t+\tau}(B/B_t)$  for a sample account ( $B_0 = \$884.13$ , FICO = 555) together with account value  $V_{t,t+\tau}(a)/B_t$  (dashed line), for  $\tau \leq 42$  days.

The DCS varies with time  $t$ . In the study period  $[0, 112]$  days, the account in Fig. 2 made two repayments, \$160.77 (18.2%) at  $T_1 = 22$  days and \$150 (20%) at  $T_2 = 53$  days. In Fig. 2a, the payback probability  $\Pi_{t,t+15 \text{ days}}(40\%)$  for repayment of at least 40% of the remaining

outstanding balance over the next  $\tau = 15$  days, conditional on the information available up to the current time  $t = 35$  days, is approximately 10%. In Fig. 2b, this probability, based on information of up to  $t = 70$  days, has increased to approximately 15%. This is primarily due to receiving a repayment (at  $T_2 = 53$ ) which updates both the account holder’s willingness-to-repay and ability-to-repay. The dotted lines indicate the expected values  $V_{t,t+\tau}$ , normalized to the interval  $[0, 1]$ , for  $t \in \{35, 70\}$  days and  $\tau \leq 45$  days. The DCS not only incorporates real-time information, it is also flexible with respect to variations of prediction horizon and payback threshold, the desired values of which can be readily substituted. Thus, at each time in the collections process the collections manager obtains a dynamic account valuation based on all available information.

### 5.2.3 Prediction Performance

The predictive accuracy of the DCS, for both MLE- and GMM-identified model parameters, is illustrated by the ROC curves in Fig. 3, where account value is used for ranking based on collectability. The ROC curves for the BIS (dotted) and the uBIS (dashed) serve as baseline reference. The true-positive rate corresponds to the percentage of payers who are identified correctly by each model. A “payer” denotes an account with at least one observed payment in the interval  $(t, t + \tau]$  with  $\tau = 45$  days. As shown in Fig. 3, the DCS performs significantly better than both BIS and uBIS, even though it uses considerably less information. Table 3d provides the area under the ROC curves (AUC) for each classifier, equivalent to the probability that a randomly chosen positive instance is ranked above a randomly chosen negative instance.

### 5.2.4 Treatment Optimization

Because the dynamic repayment model includes account-treatment actions, it is possible to use the account valuation in Prop. 3, as a function of the action process  $a_t = [a_{t,1}, a_{t,2}]^\top$ , for treatment optimization. More specifically, the bank’s net revenues depend on the account-specific commission rates  $c_m$  in the collection phases  $m \in \{1, 2\}$ . We limit attention here to the optimization of the commission rate  $a_{t,2}$ , represented by  $(c_1, c_2)$ .

A typical account in the dataset has at time  $\theta_1 = 0$  a constant commission rate of  $c_1 = 15\%$  in phase 1 and is moved after  $\theta_2 = 120$  days without full payment to phase 2 of the collections process at  $c_2 = 25\%$  commission. We compare these modal rates with optimized model predictions, given a capital cost of  $\rho = 10\%$ . The net value, over the time interval  $(t, t + \tau]$  with  $t = 0$  and  $\tau = 360$  days, as a function of FICO score and outstanding balance, is shown in Fig. 4a; one observes that, per outstanding dollar, accounts with lower balances are more valuable. Furthermore, account holders with higher pre-default FICO scores are more likely to pay, so accounts with both a high FICO score and a low outstanding balance are among the most collectable (on a per-dollar basis). Fig. 4b illustrates the marginal gain/loss when the commission rates go up, for each phase  $m \in \{1, 2\}$ . All of the curves are unimodal with a peak in the neighborhood of FICO = 550 (not shown for two curves). Thus, the marginal net yield increases in the FICO score until account quality is sufficiently high, so the price paid for additional collection effort no longer warrants the difference in collection yield.

Fig. 5a shows the net revenue of an account ( $B_0 = \$1,000$ , FICO = 600) as a function of

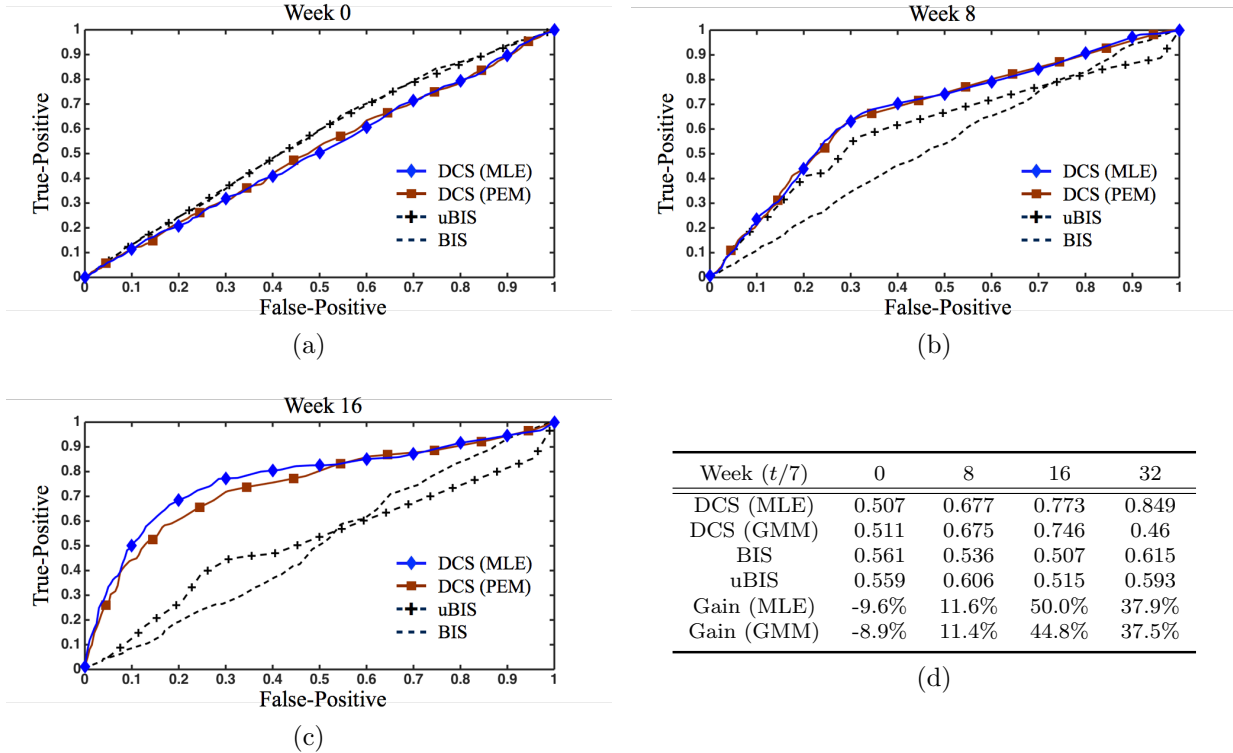


Figure 3: Scoring performance: (a)–(c) ROC curves at different times in the collections process; (d) AUC.

the commission rates  $c_2 \geq c_1$ , with maximum at  $(c_1^*, c_2^*) = (30\%, 30\%)$ . The symmetry of the commission structure follows from the underlying simplification that the agencies are homogenous. Indeed, to avoid initially overpaying for collection effort and a premature termination of the process, necessarily  $c_2^* \geq c_1^*$ . Because the two collection agencies are homogenous, the phase-2 commission  $c_2$  cannot be strictly larger than the phase-1 commission  $c_1$  at the optimum.<sup>26</sup>

Based on this, it is possible to optimally price an account for leasing it out to collection agencies. Let  $c = c_1 = c_2$  denote a symmetric commission structure. The optimal commission rate  $c^*$  can be obtained analogously to the standard monopoly pricing rule (see, e.g., Tirole 1988) by viewing  $p = 1 - c$  as the price (i.e., unit revenue) for an account leased to a collection agency. Since the outstanding balance is sunk at the time of placement, the opportunity cost of collection is zero, and the realized value (i.e., revenue) for the bank becomes  $(1 - c)V_{0,\tau}(a)$ . Given the price elasticity of the collection value,

$$\varepsilon(c) = -\frac{1 - c}{V_{0,\tau}(a)} \frac{\partial V_{0,\tau}(a)}{\partial(1 - c)} = -\frac{p}{V_{0,\tau}(a)} \frac{\partial V_{0,\tau}(a)}{\partial p},$$

we obtain that

$$\varepsilon(c^*) = 1 \tag{16}$$

<sup>26</sup>In practice, the agency in the second phase is often more specialized (e.g., a law firm) and therefore requires a higher commission rate because of its higher collection cost.

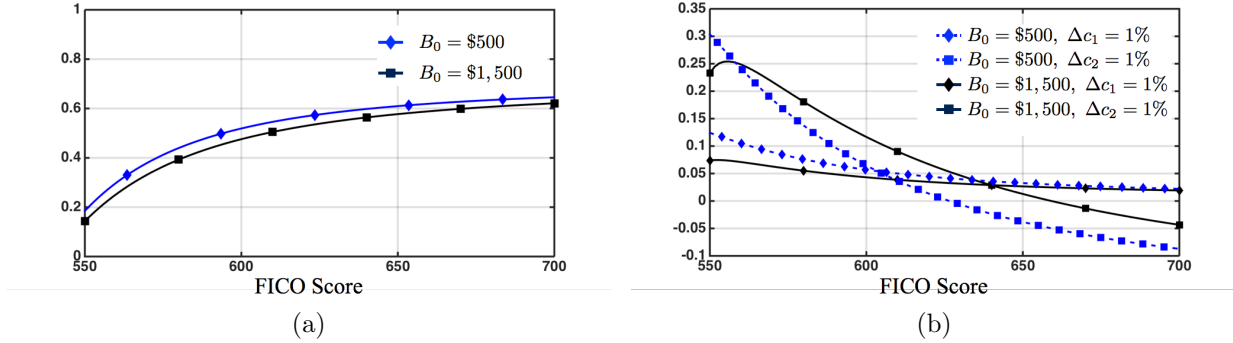


Figure 4: (a) Net revenue and (b) expected gain from commission increase, as a function of FICO for  $B_0 \in \{\$500, \$1,500\}$ , given an initial commission structure of  $(c_1, c_2) = (15\%, 25\%)$ .

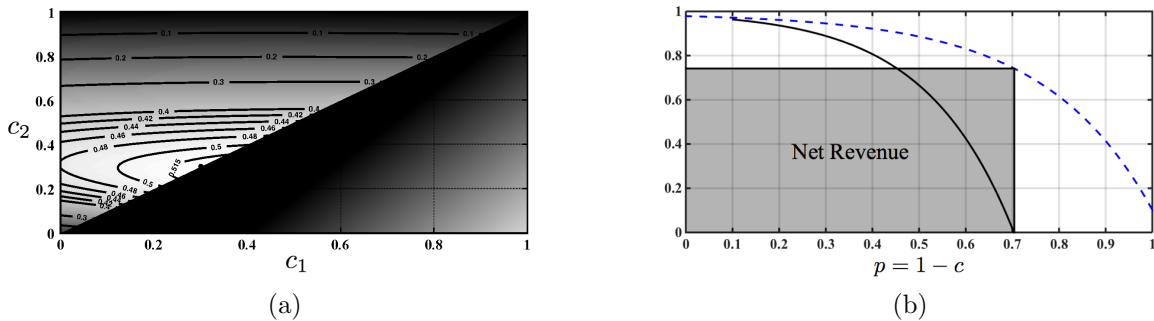


Figure 5: (a) Net revenue as a function of the commission structure  $(c_1, c_2)$ ; (b) Unit revenue  $V_{0,\tau}(a)/B_0$  (dashed line) and marginal net revenue (solid line) for a symmetric commission structure  $(c, c)$  in terms of the equivalent price  $p = 1 - c$ .

at the optimum. It is therefore best for the bank to charge a commission rate that sets the price elasticity of the collection value equals 1. Fig. 5b illustrates the monopoly pricing rule (16), implying a revenue-maximizing price of  $p^* = 1 - c^* = 70\%$  in the example.

## 6 Conclusion

The dynamic collectability score (DCS) method developed in this paper can be viewed as the backbone of collections optimization. In contrast to standard bank-internal scores, the DCS reflects the actual repayment probability conditional on a repayment threshold, a collection horizon, and scheduled account-treatment actions. The DCS is therefore a very flexible measure which can be tailored for a range of applications, such as the following six examples. First, it can be used in its base version as a forecasting and scoring tool, which by itself outperforms BIS methods in terms of the classic “payer” versus “non-payer” classification, as shown in Sec. 5. Second, the DCS figures prominently in the valuation formula of Prop. 3. Third, because the DCS is determined conditional on an action sequence, it can be used to optimize the collections process. The key difference from standard methods is that our model allows for a simple forward simulation of repayment processes. The details of the

associated stochastic optimization problem are discussed in a companion paper (Chehrazi et al. 2014). Fourth, the DCS valuation formula can be used to optimize compensation for collection agencies in terms of their optimal commission rate, at the account level. If the commission is tied to the DCS, then, because of the increased efficiency in allocating effort, both the collection agency and the credit-card issuing bank stand to gain. Fifth, because of the dynamic nature of the DCS, it is instrumental in the decision of when and how to settle a credit-card account. Last but not least, more resolution and reliability in the valuation of delinquent accounts also improves the life-cycle assessment of account value, and thus ultimately enhances the quality of a bank's underwriting decisions. Hence, by backward induction, a functioning DCS method can be expected to impact the entire account life-cycle.

## Appendix A: Analytical Details

### Feasibility Domain.

Let  $\mu(x) \equiv \mu_0 + \mu_1 x$  with  $(\mu_0, \mu_1) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x \times n_x}$ ,  $\sigma(x) \equiv \Sigma v(x)$  with  $\Sigma \in \mathbb{R}^{n_x \times n_x}$  nonsingular (Duffie and Kan 1996), and  $v(x) \equiv \text{diag}(u_1^{1/2}(x), \dots, u_{n_x}^{1/2}(x))$  with

$$u_l(x) \equiv u_{l,0} + u_{l,1}^\top x,$$

where  $u_{l,0} \in \mathbb{R}$  and  $u_{l,1} \in \mathbb{R}^{n_x}$ , for all  $l \in \{1, \dots, n_x\}$ . The domain  $\mathcal{D}$  of  $X_t$  can be described as a polyhedron,

$$\mathcal{D} = \{x \in \mathbb{R}^{n_x} : \min\{u_1(x), \dots, u_{n_x}(x)\} \geq 0\}.$$

Setting  $\lambda_\infty(x) \equiv \lambda_{\infty,0} + \lambda_{\infty,1}^\top x$ , and denoting

$$\mathbf{u}_0 = \begin{bmatrix} u_{1,0} \\ \vdots \\ u_{n_x,0} \end{bmatrix} \quad \text{and} \quad \mathbf{u}_1 = \begin{bmatrix} u_{1,1}^\top \\ \vdots \\ u_{n_x,1}^\top \end{bmatrix},$$

the following result provides conditions that ensure the positivity of the intensity process  $\lambda_t$ .

**Proposition A1** (Positivity of Intensity). *Given initial data  $(\lambda_0, X_0)$  in  $\mathbb{R}_{++} \times \mathcal{D}$ , the intensity process  $\lambda_t$  is positive<sup>27</sup> if and only if there exists a vector  $\gamma \in \mathbb{R}_+^{n_x}$  such that  $\lambda_{\infty,0} \geq \gamma^\top \mathbf{u}_0$  and  $\lambda_{\infty,1}^\top = \gamma^\top \mathbf{u}_1$ .*

**Proof.** To ensure the positivity of  $\lambda_t$  it is necessary and sufficient to ensure that  $d\lambda_t/dt$  is positive at the boundary, where  $\lambda_t = 0$ . The latter holds if and only if  $\lambda_\infty(x) \geq 0$  on  $\mathcal{D}$ , given that  $\kappa, \delta_1, \delta_2 \geq 0$  (as required by the model). We therefore need to show that

$$\lambda_\infty(x) = \lambda_{\infty,0} + \lambda_{\infty,1}^\top x \geq 0, \quad \forall x \in \mathcal{D} = \{x \in \mathbb{R}^{n_x} : \mathbf{u}_0 + \mathbf{u}_1 x \geq 0\},$$

if and only if  $\mathcal{G} = \{\gamma \in \mathbb{R}_+^{n_x} : \lambda_{\infty,0} \geq \gamma^\top \mathbf{u}_0 \text{ and } \lambda_{\infty,1}^\top = \gamma^\top \mathbf{u}_1\} \neq \emptyset$ .

*Sufficiency:* If there exists  $\gamma \in \mathcal{G}$ , then

$$\lambda(x) = \lambda_{\infty,0} + \lambda_{\infty,1}^\top x = \lambda_{\infty,0} + \gamma^\top \mathbf{u}_1 x > \gamma^\top (\mathbf{u}_0 + \mathbf{u}_1 x) \geq 0,$$

<sup>27</sup>Positivity is required only almost everywhere, i.e., everywhere except for a zero-measure set of times.



for any  $x \in \mathcal{D}$ .

*Necessity:* Assume  $(\lambda_{\infty,0}, \lambda_{\infty,1})$  is such that  $\lambda_{\infty}(x) \geq 0$  for any  $x \in \mathcal{D}$ . We now show that  $\mathcal{G}$  is nonempty. To simplify the exposition, we restrict attention to the case where  $\mathbf{u}_1$  has full rank.<sup>28</sup> The following intermediate result implies the nonemptiness of  $\mathcal{G}$ .

*Claim:* *There exists  $\gamma \in \mathbb{R}_+^{n_x}$  such that  $\lambda_{\infty,1}^\top = \gamma^\top \mathbf{u}_1$ .* *Proof:* Since  $\mathbf{u}_1$  has full rank, there exists  $\gamma \in \mathbb{R}^{n_x}$  such that  $\lambda_{\infty,1}^\top = \gamma^\top \mathbf{u}_1$ . It is therefore enough to show that  $\gamma \geq 0$ . If this is not true, then there is a unit vector  $e_l$  in  $\mathbb{R}^{n_x}$  with  $\gamma^\top e_l < 0$ . Using the Gram-Schmidt process if necessary, it is possible to find a vector  $x_l \in \mathbb{R}^{n_x}$  such that  $\mathbf{u}_1 x_l = e_l$ . For any scalar  $\varphi > 0$  and  $x \in \mathcal{D}$ , one can then conclude  $x + \varphi x_l \in \mathcal{D}$  while at the same time for  $\varphi$  large enough  $\lambda_{\infty}(x + \varphi x_l) = \lambda_{\infty,0} + \lambda_{\infty,1}^\top x + \varphi \gamma^\top e_l$  becomes negative. This yields a contradiction, and thus establishes the validity of the claim.

Since  $\mathbf{u}_1$  has full rank, there exists  $\hat{x} \in \mathcal{D}$  such that  $\mathbf{u}_0 + \mathbf{u}_1 \hat{x} = 0$ . Using the claim, there exists  $\gamma \in \mathbb{R}_+^{n_x}$  such that  $\lambda_{\infty,1}^\top = \gamma^\top \mathbf{u}_1$ , whence

$$0 \leq \lambda_{\infty}(\hat{x}) = \lambda_{\infty,0} + \lambda_{\infty,1}^\top \hat{x} = \lambda_{\infty,0} + \gamma^\top \mathbf{u}_1 \hat{x} = \lambda_{\infty,0} - \gamma^\top \mathbf{u}_0.$$

Thus, one also obtains  $\lambda_{\infty,0} \geq \gamma^\top \mathbf{u}_0$ , which implies that  $\gamma \in \mathcal{G} \neq \emptyset$ , completing our proof. ■

## Affine Jump-Diffusion Processes with Deterministic Jumps.

Consider the general case

$$d\hat{X}_t = \hat{\mu}(\hat{X}_t)dt + \hat{\sigma}(\hat{X}_t)d\hat{W}_t + \hat{\delta}_1 d\hat{J}_t + \hat{\delta}_2 d\hat{a}_t, \quad (17)$$

where  $\hat{W}_t$  is a standard Wiener process in  $\mathbb{R}^{n_{\hat{x}}}$ ;  $\hat{\mu}(\hat{x}) = \hat{\mu}_0 + \hat{\mu}_1 \hat{x}$  in which  $(\hat{\mu}_0, \hat{\mu}_1) \in \mathbb{R}^{n_{\hat{x}}} \times \mathbb{R}^{n_{\hat{x}} \times n_{\hat{x}}}$ ;  $\hat{\sigma}(\hat{x})\hat{\sigma}^\top(\hat{x}) = \hat{\sigma}_0 + \sum_{l=1}^{n_{\hat{x}}} \hat{\sigma}_1^l \hat{x}_l$  where  $(\hat{\sigma}_0, (\hat{\sigma}_1^l)_{l=1}^{n_{\hat{x}}}) \in \mathbb{R}^{n_{\hat{x}} \times n_{\hat{x}}} \times \mathbb{R}^{n_{\hat{x}} \times n_{\hat{x}} \times n_{\hat{x}}}$ ;  $\hat{J}$  is an  $\mathbb{R}^{n_j}$ -dimensional pure jump process whose jumps  $\Delta \hat{J}$  are i.i.d., and its arrival intensity is an affine function of  $\hat{X}$ , of the form  $\hat{\lambda}(\hat{X}_t) = \hat{\lambda}_0 + \hat{\lambda}_1^\top \hat{X}_t$  with  $(\hat{\lambda}_0, \hat{\lambda}_1) \in \mathbb{R} \times \mathbb{R}^{n_{\hat{x}}}$ ;  $\hat{a}$  is a deterministic, right-continuous jump process, whose jumps are at fixed, increasing arrival times  $\hat{\theta}_m$ , for  $m \geq 1$ ; and  $(\hat{\delta}_1, \hat{\delta}_2) \in \mathbb{R}^{n_{\hat{x}} \times n_j} \times \mathbb{R}^{n_{\hat{x}} \times n_a}$ . The following proposition provides a semi-analytic expression for the transform

$$\mathcal{L}_{(t,t+\tau]}(\hat{\rho}_0, \hat{\rho}_1, \hat{\zeta}, \hat{X}) \triangleq \mathbb{E}(e^{-\int_t^{t+\tau} \hat{\rho}(\hat{X}_s) ds + \hat{\zeta}^\top \hat{X}_{t+\tau}} | \mathcal{F}_t), \quad (18)$$

where  $\hat{\zeta} \in \mathbb{C}^{n_{\hat{x}}}$ ,  $\hat{\rho}(\hat{x}) = \hat{\rho}_0 + \hat{\rho}_1^\top \hat{x}$  with  $\hat{\rho}_0 \in \mathbb{R}$ ,  $\hat{\rho}_1 \in \mathbb{R}^{n_{\hat{x}}}$ .

**Proposition A2** (Affine Transform). *Let  $\hat{\theta}_m$ ,  $m \in \{\underline{m}, \dots, \bar{m}\}$ , denote the jump arrival times of  $\hat{a}_s$ , for  $s \in (t, t + \tau]$ , and let  $(\hat{\alpha}, \hat{\beta})$  be the (unique) solution to the initial-value problem*

$$\dot{\hat{\alpha}}_s + \hat{\mu}_0^\top \hat{\beta}_s + \hat{\beta}_s^\top \hat{\sigma}_0 \hat{\sigma}_0^\top \hat{\beta}_s / 2 = \hat{\rho}_0 + (1 - \mathbb{E}[e^{\hat{\beta}_s^\top \hat{\delta}_1 \Delta \hat{J}}]) \hat{\lambda}_0, \quad \hat{\alpha}(t + \tau) = 0, \quad (19)$$

$$\dot{\hat{\beta}}_s + \hat{\mu}_1^\top \hat{\beta}_s + \hat{\beta}_s^\top \hat{\sigma}_1 \hat{\sigma}_1^\top \hat{\beta}_s / 2 = \hat{\rho}_1 + (1 - \mathbb{E}[e^{\hat{\beta}_s^\top \hat{\delta}_1 \Delta \hat{J}}]) \hat{\lambda}_1, \quad \hat{\beta}(t + \tau) = \hat{\zeta}, \quad (20)$$

on  $[t, t + \tau]$ . *If the conditions*

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<sup>28</sup>A complete proof is available upon request. Important for the results in this paper is the sufficiency of  $\mathcal{G} \neq \emptyset$  for the positivity of  $\lambda_t$ . The necessity provides a guarantee that the nonemptiness of  $\mathcal{G}$  constitutes a minimal requirement for the admissibility of the intensity process.

$$(i) \mathbb{E} \left[ \int_0^{t+\tau} \|\hat{\xi}_s\| ds \right] < \infty, \text{ with } \hat{\xi}_s \triangleq \hat{\lambda}(\hat{X}_s) \hat{\Psi}_s (\mathbb{E}[e^{\hat{\beta}_s^\top \hat{\delta}_1 \Delta \hat{J}}] - 1),$$

$$(ii) \mathbb{E} \left[ \left( \int_0^{t+\tau} \|\hat{\eta}_s\|^2 ds \right)^{1/2} \right] < \infty, \text{ with } \hat{\eta}_s \triangleq \hat{\Psi}_s \hat{\beta}_s^\top \hat{\sigma}(\hat{X}_s),$$

$$(iii) \mathbb{E} \left[ \|\hat{\Psi}_{t+\tau}\| \right] < \infty, \text{ with } \hat{\Psi}_s \triangleq \exp \left[ \hat{\alpha}_s + \hat{\beta}_s^\top \hat{X}_s - \int_0^s \hat{\rho}(\hat{X}_s) ds \right],$$

hold, then

$$\mathcal{L}_{(t,t+\tau]}(\hat{\rho}_0, \hat{\rho}_1, \hat{\zeta}, \hat{X}) = \exp \left[ \hat{\alpha}_t + \hat{\beta}_t^\top \hat{X}_t + \sum_{m=\bar{m}}^{\bar{m}} \hat{\beta}_{\hat{\theta}_m}^\top \hat{\delta}_2 \Delta \hat{a}_{\hat{\theta}_m} \right],$$

where  $\Delta \hat{a}_s \triangleq \hat{a}_s - \hat{a}_{s-}$ .

**Proof.** Using the law of iterated expectations, Eq. (18) becomes

$$\mathbb{E} \left[ e^{-\int_t^{t+\tau} \hat{\rho}(\hat{X}_s) ds + \hat{\zeta}^\top \hat{X}_{t+\tau}} \middle| \mathcal{F}_t \right] = \mathbb{E} \left[ e^{-\int_t^{\hat{\theta}_{\bar{m}}} \hat{\rho}(\hat{X}_s) ds} \mathbb{E} \left[ e^{-\int_{\hat{\theta}_{\bar{m}}}^{t+\tau} \hat{\rho}(\hat{X}_s) ds + \hat{\zeta}^\top \hat{X}_{t+\tau}} \middle| \mathcal{F}_{\hat{\theta}_{\bar{m}}} \right] \middle| \mathcal{F}_t \right].$$

For  $s \in [\hat{\theta}_{\bar{m}}, t + \tau]$ , the process  $\hat{a}_s$  is constant, so  $\hat{X}_s$  exhibits affine jump-diffusion dynamics. Thus, by Prop. 1 of Duffie et al. (2000),<sup>29</sup>

$$\mathbb{E} \left[ e^{-\int_t^{\hat{\theta}_{\bar{m}}} \hat{\rho}(\hat{X}_s) ds} \mathbb{E} \left[ e^{-\int_{\hat{\theta}_{\bar{m}}}^{t+\tau} \hat{\rho}(\hat{X}_s) ds + \hat{\zeta}^\top \hat{X}_{t+\tau}} \middle| \mathcal{F}_{\hat{\theta}_{\bar{m}}} \right] \middle| \mathcal{F}_t \right] = \mathbb{E} \left[ e^{-\int_t^{\hat{\theta}_{\bar{m}}} \hat{\rho}(\hat{X}_s) ds + \hat{\alpha}(\hat{\theta}_{\bar{m}}) + \hat{\beta}_{\hat{\theta}_{\bar{m}}}^\top \hat{X}_{\hat{\theta}_{\bar{m}}}} \middle| \mathcal{F}_t \right] \quad (21)$$

where for  $s \in [\hat{\theta}_{\bar{m}}, t + \tau]$ :

$$\begin{aligned} \dot{\hat{\alpha}}_s + \hat{\mu}_0^\top \hat{\beta}_s + \hat{\beta}_s^\top \hat{\sigma}_0 \hat{\sigma}_0^\top \hat{\beta}_s / 2 &= \hat{\rho}_0 + (1 - \mathbb{E}[e^{\hat{\beta}_s^\top \hat{\delta}_1 \Delta \hat{J}}]) \hat{\lambda}_0, & \hat{\alpha}_{t+\tau} &= 0, \\ \dot{\hat{\beta}}_s + \hat{\mu}_1^\top \hat{\beta}_s + \hat{\beta}_s^\top \hat{\sigma}_1 \hat{\sigma}_1^\top \hat{\beta}_s / 2 &= \hat{\rho}_1 + (1 - \mathbb{E}[e^{\hat{\beta}_s^\top \hat{\delta}_1 \Delta \hat{J}}]) \hat{\lambda}_1, & \hat{\beta}_{t+\tau} &= \hat{\zeta}. \end{aligned}$$

Rewriting Eq. (21) in the form

$$\begin{aligned} \mathbb{E} \left[ e^{-\int_t^{\hat{\theta}_{\bar{m}}} \hat{\rho}(\hat{X}_s) ds + \hat{\alpha}_{\hat{\theta}_{\bar{m}}} + \hat{\beta}_{\hat{\theta}_{\bar{m}}}^\top \hat{X}_{\hat{\theta}_{\bar{m}}}} \middle| \mathcal{F}_t \right] &= \mathbb{E} \left[ e^{-\int_t^{\hat{\theta}_{\bar{m}}} \hat{\rho}(\hat{X}_s) ds + \hat{\alpha}_{\hat{\theta}_{\bar{m}}} + \hat{\beta}_{\hat{\theta}_{\bar{m}}}^\top (\hat{X}_{\hat{\theta}_{\bar{m}}} - \hat{\delta}_2 \Delta \hat{a}_{\hat{\theta}_{\bar{m}}}) + \hat{\beta}_{\hat{\theta}_{\bar{m}}}^\top \hat{\delta}_2 \Delta \hat{a}_{\hat{\theta}_{\bar{m}}}} \middle| \mathcal{F}_t \right] \\ &= e^{\hat{\alpha}_{\hat{\theta}_{\bar{m}}} + \hat{\beta}_{\hat{\theta}_{\bar{m}}}^\top \hat{\delta}_2 \Delta \hat{a}_{\hat{\theta}_{\bar{m}}}} \mathbb{E} \left[ e^{-\int_t^{\hat{\theta}_{\bar{m}}} \hat{\rho}(\hat{X}_s) ds + \hat{\beta}_{\hat{\theta}_{\bar{m}}}^\top (\hat{X}_{\hat{\theta}_{\bar{m}}} - \hat{\delta}_2 \Delta \hat{a}_{\hat{\theta}_{\bar{m}}})} \middle| \mathcal{F}_t \right], \end{aligned}$$

one obtains that the effect of the deterministic jump factor  $\hat{a}$  is decoupled from the dynamics of  $\hat{X}_s$  for  $s \in [\hat{\theta}_{\bar{m}-1}, \hat{\theta}_{\bar{m}}]$ . Rewriting  $-\int_t^{\hat{\theta}_{\bar{m}}} \hat{\rho}(\hat{X}_s) ds$  as  $-\int_t^{\hat{\theta}_{\bar{m}-1}} \hat{\rho}(\hat{X}_s) ds - \int_{\hat{\theta}_{\bar{m}-1}}^{\hat{\theta}_{\bar{m}}} \hat{\rho}(\hat{X}_s) ds$ , by using the law of iterated expectations and again Prop. 1 by Duffie et al. (2000), one obtains

$$\mathbb{E} \left[ e^{-\int_t^{t+\tau} \hat{\rho}(\hat{X}_s) ds + \hat{\zeta}^\top \hat{X}_{t+\tau}} \middle| \mathcal{F}_t \right] = e^{\hat{\alpha}_{\hat{\theta}_{\bar{m}}} + \hat{\beta}_{\hat{\theta}_{\bar{m}}}^\top \hat{\delta}_2 \Delta \hat{a}_{\hat{\theta}_{\bar{m}}}} \mathbb{E} \left[ e^{-\int_t^{\hat{\theta}_{\bar{m}-1}} \hat{\rho}(\hat{X}_s) ds + \check{\alpha}_{\hat{\theta}_{\bar{m}-1}} + \check{\beta}_{\hat{\theta}_{\bar{m}-1}}^\top \hat{X}_{\hat{\theta}_{\bar{m}-1}}} \middle| \mathcal{F}_t \right], \quad (22)$$

where  $(\check{\alpha}_s, \check{\beta}_s)$ , for all  $s \in [\hat{\theta}_{\bar{m}-1}, \hat{\theta}_{\bar{m}}]$ , is determined as solution of the initial-value problem

$$\begin{aligned} \dot{\check{\alpha}}_s + \hat{\mu}_0^\top \check{\beta}_s + \check{\beta}_s^\top \hat{\sigma}_0 \hat{\sigma}_0^\top \check{\beta}_s / 2 &= \hat{\rho}_0 + (1 - \mathbb{E}[e^{\check{\beta}_s^\top \hat{\delta}_1 \Delta \hat{J}}]) \hat{\lambda}_0, & \check{\alpha}_{\hat{\theta}_{\bar{m}}} &= 0, \\ \dot{\check{\beta}}_s + \hat{\mu}_1^\top \check{\beta}_s + \check{\beta}_s^\top \hat{\sigma}_1 \hat{\sigma}_1^\top \check{\beta}_s / 2 &= \hat{\rho}_1 + (1 - \mathbb{E}[e^{\check{\beta}_s^\top \hat{\delta}_1 \Delta \hat{J}}]) \hat{\lambda}_1, & \check{\beta}_{\hat{\theta}_{\bar{m}}} &= \hat{\beta}_{\hat{\theta}_{\bar{m}}}. \end{aligned}$$

<sup>29</sup>The result remains valid for  $(\hat{\rho}_0, \hat{\rho}_1) \in \mathbb{C} \times \mathbb{C}^{n_x}$ .

Since the ODEs governing the dynamics of  $\hat{\alpha}_s$  and  $\hat{\beta}_s$  on  $[\hat{\theta}_{\bar{m}}, t+\tau]$  and  $\check{\alpha}_s$  and  $\check{\beta}_s$  on  $[\hat{\theta}_{\bar{m}-1}, \hat{\theta}_{\bar{m}}]$  are identical, Eq. (22) simplifies to

$$\mathbb{E} \left[ e^{-\int_t^{t+\tau} \hat{\rho}(\hat{X}_s) ds + \hat{\zeta}^\top \hat{X}_{t+\tau}} \middle| \mathcal{F}_t \right] = e^{\hat{\alpha}_{\hat{\theta}_{\bar{m}-1}} + \sum_{m=\bar{m}-1}^{\bar{m}} \hat{\beta}_m^\top \hat{\delta}_2 \Delta \hat{\alpha}_{\hat{\theta}_m}} \times \mathbb{E} \left[ e^{-\int_t^{\hat{\theta}_{\bar{m}-1}} \hat{\rho}(\hat{X}_s) ds + \hat{\beta}_{\hat{\theta}_{\bar{m}-1}}^\top (\hat{X}_{\hat{\theta}_{\bar{m}-1}} - \hat{\delta}_2 \Delta \hat{\alpha}_{\hat{\theta}_{\bar{m}-1}})} \middle| \mathcal{F}_t \right],$$

where  $(\hat{\alpha}_s, \hat{\beta}_s)$  solves the initial-value problem

$$\dot{\hat{\alpha}}_s + \hat{\mu}_0^\top \hat{\beta}_s + \hat{\beta}_s^\top \hat{\sigma}_0 \hat{\sigma}_0^\top \hat{\beta}_s / 2 = \hat{\rho}_0 + (1 - \mathbb{E}[e^{\hat{\beta}_s^\top \hat{\delta}_1 \Delta \hat{J}}]) \hat{\lambda}_0, \quad \hat{\alpha}_{t+\tau} = 0, \quad (23)$$

$$\dot{\hat{\beta}}_s + \hat{\mu}_1^\top \hat{\beta}_s + \hat{\beta}_s^\top \hat{\sigma}_1 \hat{\sigma}_1^\top \hat{\beta}_s / 2 = \hat{\rho}_1 + (1 - \mathbb{E}[e^{\hat{\beta}_s^\top \hat{\delta}_1 \Delta \hat{J}}]) \hat{\lambda}_1, \quad \hat{\beta}_{t+\tau} = \hat{\zeta}, \quad (24)$$

for all  $s \in [\hat{\theta}_{\bar{m}-1}, t+\tau]$ . Continuing the extension towards the left in the same manner yields

$$\mathbb{E} \left[ e^{-\int_t^{t+\tau} \hat{\rho}(\hat{X}_s) ds + \hat{\zeta}^\top \hat{X}_{t+\tau}} \middle| \mathcal{F}_t \right] = e^{\hat{\alpha}_t + \hat{\beta}_t^\top \hat{X}_t + \sum_{m=m}^{\bar{m}} \hat{\beta}_m^\top \hat{\delta}_2 \Delta \hat{\alpha}_{\hat{\theta}_m}},$$

where  $(\hat{\alpha}, \hat{\beta})$  solves the initial-value problem (23)–(24) on  $[t, t+\tau]$ . ■

**Proof of Proposition 1.** The proof follows from the proof of Prop. A2. Specifically, let  $\hat{X}_t = [\mathcal{Q}_t, \lambda_t, X_t^\top]^\top$ , where

$$\begin{aligned} d\mathcal{Q}_t &= d\mathcal{Q}_t, \\ d\lambda_t &= \kappa(\lambda_\infty(X_t) - \lambda_t)dt + \delta_1^\top dJ_t + \delta_2^\top da_t, \\ dX_t &= \mu(X_t)dt + \sigma(X_t)dW_t. \end{aligned}$$

Then (referring to Table B1 in Appendix B for any extra notational definition) it is

$$\hat{\mu}_0 = \begin{bmatrix} 0 \\ \kappa \lambda_{\infty,0} \\ \mu_0 \end{bmatrix} \in \mathbb{R}^{2+n_X}, \quad \hat{\mu}_1 = \left[ \begin{array}{c|c|c} 0 & 0 & 0 \\ 0 & -\kappa & \kappa \lambda_{\infty,1}^\top \\ \hline 0_{(n_X \times 2)} & & \mu_1 \end{array} \right] \in \mathbb{R}^{(2+n_X) \times (2+n_X)}, \quad (25)$$

$\hat{\lambda}_0 = 0$ ,  $\hat{\lambda}_1 = \hat{e}_2$  with  $\hat{e}_l$  the  $l$ -th standard Euclidean basis vector of  $\mathbb{R}^{2+n_X}$ , and

$$\hat{\delta}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \delta_{11} & \delta_{12} \\ \hline 0_{n_X \times 3} \end{bmatrix} \in \mathbb{R}^{(2+n_X) \times 3}, \quad \hat{\delta}_2 = \begin{bmatrix} 0_{1 \times n_a} \\ \delta_2^\top \\ \hline 0_{n_X \times n_a} \end{bmatrix} \in \mathbb{R}^{(2+n_X) \times n_a}. \quad (26)$$

Futhermore,  $\Delta \hat{J} = [Q, 1, R]^\top$  and  $\hat{\sigma}(\hat{x}) = \hat{\Sigma} \hat{v}(\hat{x})$ , with  $\hat{\Sigma}, \hat{v}(\hat{x}) \in \mathbb{R}^{(2+n_X) \times (2+n_X)}$  and

$$\hat{\Sigma} = \left[ \begin{array}{c|c} 0_{(2 \times 2)} & 0_{(2 \times n_X)} \\ \hline 0_{(n_X \times 2)} & \Sigma \end{array} \right], \quad \hat{v}(\hat{x}) = \left[ \begin{array}{c|c} 0_{(2 \times 2)} & 0_{(2 \times n_X)} \\ \hline 0_{(n_X \times 2)} & v(x) \end{array} \right], \quad (27)$$

where  $\Sigma, v(x) \in \mathbb{R}^{n_X \times n_X}$  are detailed at the outset of this appendix. Hence, one obtains

$$\begin{aligned} \mathbb{E}[e^{\zeta(\mathcal{Q}_{t+\tau} - \mathcal{Q}_t)} | \mathcal{F}_t] &= \mathcal{L}_{(t,t+\tau]}(0, 0, \zeta \hat{e}_1, \hat{X}) \cdot \mathcal{L}_{(t,t]}(0, 0, -\zeta \hat{e}_1, \hat{X}) \\ &= \exp \left[ -\Delta \hat{\alpha}_{t,t+\tau} - \Delta \hat{\beta}_{t,t+\tau}^\top \hat{X}_t + \sum_{m=m}^{\bar{m}} \hat{\beta}_{\theta_m}^\top \hat{\delta}_2 \Delta a_{\theta_m} \right], \end{aligned}$$

which concludes the proof.  $\blacksquare$

**Proof of Proposition 2.** Using the CCF of  $\mathcal{Q}_{t+\tau} - \mathcal{Q}_t$ , provided by Prop. 1, the Fourier transform of the CDF of  $\mathcal{Q}_{t+\tau} - \mathcal{Q}_t$  can be written as

$$\pi \delta^d(w) + \frac{\exp \left[ -\Delta \hat{\alpha}_{t,t+\tau}(-jw) - \Delta \hat{\beta}_{t,t+\tau}^\top(-jw) \hat{X}_t + \sum_{m=m}^{\bar{m}} \hat{\beta}_{\theta_m}^\top(-jw) \hat{\delta}_2 \Delta a_{\theta_m} \right]}{jw},$$

where  $\delta^d(\cdot)$  denotes the Dirac delta function (a distribution). Taking the inverse Fourier transform of the above equation at  $\log(1 - B/B_t)$  yields the result.  $\blacksquare$

**Proof of Proposition 3.** Let  $\rho > 0$  be a discount factor. The value of an account over  $(t, t + \tau]$  is defined as the expected sum of discounted payments,

$$V_{t,t+\tau}(a) = \mathbb{E} \left[ \sum_{i:T_i \in (t,t+\tau]} e^{-\rho(T_i-t)} Z_i \middle| \mathcal{F}_t, a \right] = \mathbb{E} \left[ \int_{(t,t+\tau] \times [0,1]} e^{-\rho(\hat{s}-t)} r B_{\hat{s}-} N(d\hat{s}, dr) \middle| \mathcal{F}_t, a \right],$$

where  $N(d\hat{s}, dr)$  is the equivalent random-measure representation of  $N = (T_i, R_i, i \geq 1)$  and  $B_{\hat{s}} = B_0 - \sum_i Z_i \mathbf{1}_{\{T_i \leq \hat{s}\}}$  denotes the remaining outstanding balance at  $\hat{s} \in (t, t + \tau]$ . Using Fubini's theorem, we obtain that

$$\mathbb{E} \left[ \int_{(t,t+\tau]} e^{-\rho(\hat{s}-t)} \int_{[0,1]} r B_{\hat{s}-} N(d\hat{s}, dr) \middle| \mathcal{F}_t, a \right] = \int_{(t,t+\tau]} e^{-\rho(\hat{s}-t)} \mathbb{E} \left[ \int_{[0,1]} r B_{\hat{s}-} N(d\hat{s}, dr) \middle| \mathcal{F}_t, a \right].$$

Furthermore,

$$\begin{aligned} \int_{(t,t+s]} \mathbb{E} \left[ \int_{[0,1]} r B_{\hat{s}-} N(d\hat{s}, dr) \middle| \mathcal{F}_t, a \right] &= \mathbb{E} \left[ \int_{(t,t+s] \times [0,1]} r B_{\hat{s}-} N(d\hat{s}, dr) \middle| \mathcal{F}_t, a \right] \\ &= \mathbb{E} \left[ \sum_{T_i \in (t,t+s]} Z_i \middle| \mathcal{F}_t, a \right] \\ &= B_t (1 - \mathbb{E}[\exp(\mathcal{Q}_{t+s} - \mathcal{Q}_t) | \mathcal{F}_t, a]). \end{aligned}$$

By Prop. 1, it is

$$\mathbb{E}[\exp(\mathcal{Q}_{t+s} - \mathcal{Q}_t) | \mathcal{F}_t, a] = \exp \left( -\Delta \hat{\alpha}_{t+\tau-s,t+\tau} - \Delta \hat{\beta}_{t+\tau-s,t+\tau}^\top \hat{X}_t + \sum_{\theta_m \in (t,t+s]} \hat{\beta}_{\tau-s+\theta_m}^\top \hat{\delta}_2^\top \Delta a_{\theta_i} \right),$$

where  $(\hat{\alpha}, \hat{\beta})$  solves the initial-value problem

$$\begin{aligned} \dot{\hat{\alpha}}_s + \hat{\mu}_0^\top \hat{\beta}_s + \hat{\beta}_s^\top \hat{\sigma}_0 \hat{\sigma}_0^\top \hat{\beta}_s / 2 &= 0, & \hat{\alpha}_{t+\tau} &= 0, \\ \dot{\hat{\beta}}_s + \hat{\mu}_1^\top \hat{\beta}_s + \hat{\beta}_s^\top \hat{\sigma}_1 \hat{\sigma}_1^\top \hat{\beta}_s / 2 &= (1 - \mathbb{E}[e^{\hat{\beta}_s^\top \hat{\delta}_1 \Delta \hat{J}}]) \hat{e}_2, & \hat{\beta}_{t+\tau} &= \hat{e}_1, \end{aligned}$$

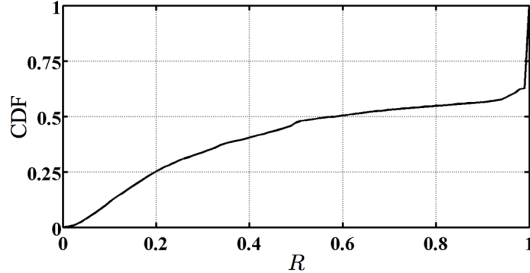


Figure A1: Empirical CDF of  $R$ .

on  $[t, t + \tau]$ . Denoting  $1 - \mathbb{E}[\exp(\mathcal{Q}_{t+s} - \mathcal{Q}_t) | \mathcal{F}_t, a]$  by  $\nu_t(s|a)$ , which is a nondecreasing right-continuous function, the account value is given by Eq. (11). ■

### Model Identification.

In our implementation of the model, in addition to the feasibility constraints of Prop. A1, a goodness-of-fit constraint is imposed to bound the parameter domain. This constraint relies on the fact that any point process  $\mathcal{N}_t$  (with  $\mathcal{N}_\infty = \infty$ ) can be transformed into a standard Poisson process using an appropriate change of time (Papangelou 1972). By imposing an upper bound for the distance of the transformed point process to a standard Poisson process one can bound  $\Theta_\lambda$ . In addition to facilitating the numerical implementation, constraining the parameter space is necessary, since taking the derivative of Eq. (13) with respect to  $\tilde{\kappa}$  and subsequently the limit as  $\tilde{\kappa} \rightarrow \infty$  yields

$$\lim_{\tilde{\kappa} \rightarrow \infty} \tilde{\kappa}^2 \frac{\partial}{\partial \tilde{\kappa}} L((T_i^k, 1 \leq i \leq \mathcal{N}_{h^k}^k) | (X_s, s \in [0, h^k]), (R_i^k, 1 \leq i \leq \mathcal{N}_{h^k}^k), \tilde{\vartheta}_\lambda) > 0,$$

so that for  $\tilde{\kappa}$  large enough the likelihood function is strictly increasing. In our implementation, we use the Cramér-von Mises goodness-of-fit measure,  $\text{CVM}((T_i^k, 1 \leq i \leq \mathcal{N}_{h^k}^k)_{k=1}^{\tilde{\kappa}}, \tilde{\vartheta}_\lambda)$ , as proximity metric (Lehmann and Romano 2005).

### Censored-Data Treatment.

In our dataset, a significant portion (32%) of accounts do not pay in full; for those the last interarrival times are generally censored over the given finite study period. This truncation phenomenon introduces a marked spike in the empirical distribution of payment interarrival times  $S_i^k = T_i^k - T_{i-1}^k$  as accounts are moved to the next collection phase following some defined period of inactivity. Consequently, any  $\tilde{\vartheta}_\lambda$  (for which the intensity process is positive) fails to satisfy the Cramér-von Mises goodness-of-fit measure at the 5%-level. This problem is resolved in our implementation by virtue of the fact that under the true intensity process  $\lambda_t(\vartheta_\lambda)$  the  $S_i^k(\vartheta_\lambda)$  are i.i.d. exponentially distributed. Thus, because exponential random variables are memoryless, the censored portion of the last interarrival payment time of each account can be reconstructed by a draw of an exponential random variable. In our implementation,  $\tilde{\vartheta}_\lambda$  is considered feasible if both the corresponding intensity process is positive *and* the CVM test does not fail (with censored data reconstructed as before).

## Appendix B: Notation

<i>Symbol</i>	<i>Description</i>	<i>Range</i>
$t$	Current time	$\mathbb{R}_+$
$i$	Index of repayment event	$\mathbb{N}$
$T_i$	Arrival time of $i$ -th repayment	$\mathbb{R}_+$
$Z_i$	Amount of $i$ -th repayment	$\mathbb{R}_+$
$B_t; B_i$	Outstanding balance (at time $t$ ; at $t = T_i$ )	$\mathbb{R}_+$
$R_i \triangleq Z_i/B_{i-1}$	Relative repayment at $t = T_i$	$[0, 1]$
$F_R$	Distribution of relative repayment $R_i$	$\mathbb{L}^\infty$
$\bar{r}$	Expected relative repayment ( $\bar{r} \triangleq \mathbb{E}[R_i]$ )	$[0, 1]$
$\mathcal{N}_t \triangleq \sum_i \mathbf{1}_{\{T_i \leq t\}}$	Repayment counting process (willingness-to-repay)	$\mathbb{N}$
$\mathcal{R}_t \triangleq \sum_i R_i \mathbf{1}_{\{T_i \leq t\}}$	Cumulative relative repayment process (ability-to-repay)	$\mathbb{R}_+$
$Q_i \triangleq \log(1 - R_i)$	Normalized logarithmic balance at time $t = T_i$	$\mathbb{R}_-$
$F_Q$	Distribution of normalized logarithmic balance $Q_i$	$\mathbb{L}^\infty$
$\mathcal{Q}_t \triangleq \sum_i Q_i \mathbf{1}_{\{T_i \leq t\}}$	Cumulative normalized logarithmic balance process	$\mathbb{R}_-$
$J_t \triangleq [\mathcal{N}_t, \mathcal{R}_t]^\top$	Repayment process	$\mathbb{N} \times \mathbb{R}_+$
$a_t$	Account-treatment schedule	$\mathbb{R}_+^{n_a}$
$m$	Index of account-treatment action	$\mathbb{N}_+$
$\theta_m$	Time of $m$ -th scheduled treatment action	$\mathbb{R}_+$
$\lambda_t$	Intensity process	$\mathbb{R}_+$
$y$	Account-specific covariates	$\mathbb{R}^{n_y}$
$\lambda_\infty(x) \triangleq \lambda_{\infty,0} + \lambda_{\infty,1}^\top x$	Long-run steady-state intensity	$\mathbb{R}_+$
$\kappa(y) \triangleq \kappa_0 + \kappa_1 y$	Mean reversion rate for intensity process	$\mathbb{R}_+$
$\delta_1(y) \triangleq [\delta_{11}(y), \delta_{12}(y)]^\top$	Sensitivity of $\lambda_t$ to $J_t$	$\mathbb{R}_+^2$
$\delta_2(y)$	Sensitivity of $\lambda_t$ to $a_t$	$\mathbb{R}_+^{n_a}$
$W_t$	Wiener process	$\mathbb{R}^{n_x}$
$X_t$	Group-covariate process	$\mathbb{R}^{n_x}$
$\mu(x) \triangleq \mu_0 + \mu_1 x$	Drift term of group covariates	$\mathbb{R}^{n_x}$
$\sigma\sigma^\top(x) \triangleq \sigma_0 + \sum_{l=1}^{n_x} \sigma_1^l x_l$	Diffusion term of group covariates	$\mathbb{R}_+^{n_x \times n_x}$
$\mathbb{F} \triangleq \{\mathcal{F}_t : t \in \mathbb{R}_+\}$	Information filtration	
$\mathcal{F}_t$	Available information at $t$	$\mathbb{F}$
$e_l$	Unit vector in the standard Euclidean base of $\mathbb{R}^{n_x}$	$\{e_1, \dots, e_{n_x}\}$
$\hat{e}_l$	Unit vector in the standard Euclidean base of $\mathbb{R}^{n_{\hat{x}}}$	$\{e_1, \dots, e_{n_{\hat{x}}}\}$
$j$	Imaginary unit	$\{\sqrt{-1}\}$
$s$	Generic time index	$\mathbb{R}_+$
$(\alpha_s, \beta_s)$	Solution of the ODE (6)–(7)	$\mathbb{C}^2$

Table B1: Summary of main notation, in the order of appearance.

<i>Symbol</i>	<i>Description</i>	<i>Range</i>
$\tau$	Prediction horizon	$\mathbb{R}_{++}$
$\nu_t(s a)$	Cash-flow measure	$[0, 1]$
$\rho$	Discount factor	$\mathbb{R}_+$
$V_{t,t+\tau}$	Expected $\tau$ -horizon present value	$\mathbb{R}_+$
$v_{t,t+\tau}$	Realized $\tau$ -horizon present value	$\mathbb{R}_+$
$\Pi_{t,t+\tau}$	Repayment probability on $(t, t + \tau]$ (DCS)	$[0, 1]$
$\mathcal{K} \triangleq \{1, \dots, \bar{k}\}$	Account portfolio	$2^{\mathcal{N}}$
$k$	Index of account (in portfolio $\mathcal{K} = \{1, \dots, \bar{k}\}$ )	$\mathcal{K}$
$h^k$	Study horizon for account $k$	$\mathbb{R}_{++}$
$\vartheta \triangleq [\vartheta_\lambda; \vartheta_X]$	Model-parameter tuple	$\Theta_\lambda \times \Theta_X$
$\varepsilon$	Price elasticity of collection value	$\mathbb{R}$

Table B1: Summary of main notation, in the order of appearance (cont'd).

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